

## $\delta_{\sim_2}^1$ WITHOUT SHARPS

SY D. FRIEDMAN AND W. HUGH WOODIN

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ABSTRACT. We show that the supremum of the lengths of  $\Delta_{\sim_2}^1$  prewellorderings of the reals can be  $\omega_2$ , with  $\omega_1$  inaccessible to reals, assuming only the consistency of an inaccessible.

$\delta_{\sim_2}^1$  denotes the supremum of the lengths of  $\Delta_{\sim_2}^1$  prewellorderings of the reals. A result of Kunen and Martin (see Martin [77]) states that  $\delta_{\sim_2}^1$  is at most  $\omega_2$ , and it is known that in the presence of sharps the assumption  $\delta_{\sim_2}^1 = \omega_2$  is strong: it implies the consistency of a strong cardinal (see Steel-Welch [?]).

In this paper we show how to obtain the consistency of  $\delta_{\sim_2}^1 = \omega_2$  in the absence of sharps, without strong assumptions.

**Theorem.** *Assume the consistency of an inaccessible. Then it is consistent that  $\delta_{\sim_2}^1 = \omega_2$  and  $\omega_1$  is inaccessible to reals (i.e.,  $\omega_1^{L[x]}$  is countable for each real  $x$ ).*

The proof is obtained by combining the  $\Delta_1$ -coding technique of Friedman-Velickovic [96] with the use of a product of Jensen codings of Friedman [94].

We begin with a description of the  $\Delta_1$ -coding technique.

**Definitions.** Suppose  $x$  is a set,  $\langle x, \epsilon \rangle$  satisfies the axiom of extensionality and  $A \subseteq \text{ORD}$ .  $x$  preserves  $A$  if  $\langle x, A \cap x \rangle \cong \langle \bar{x}, A \cap \bar{x} \rangle$  where  $\bar{x}$  = transitive collapse of  $x$ . For any ordinal  $\delta$ ,  $x[\delta] = \{f(\gamma) \mid \gamma < \delta, f \in x, f \text{ a function}, \gamma \in \text{Dom}(f)\}$ .  $x$  strongly preserves  $A$  if  $x[\delta]$  preserves  $A$  for every cardinal  $\delta$ . A sequence  $x_0 \subseteq x_1 \subseteq \dots$  is tight if it is continuous and for each  $i$ ,  $\langle \bar{x}_j \mid j < i \rangle$  belongs to the least  $ZF^-$ -model which contains  $\bar{x}_i$  as an element and correctly computes  $\text{card}(\bar{x}_i)$ .

**Condensation Condition for A.** Suppose  $t$  is transitive,  $\delta$  is regular,  $\delta \in t$  and  $x \in t$ . Then:

- (a) There exists a continuous, tight  $\delta$ -sequence  $x_0 \prec x_1 \prec \dots \prec t$  such that  $\text{card}(x_i) = \delta$ ,  $x \in x_0$  and  $x_i$  strongly preserves  $A$ , for each  $i$ .
- (b) If  $\delta$  is inaccessible, then there exist  $x_i$ 's as above but where  $\text{card}(x_i) = \aleph_i$ .

The following is proved in Friedman-Velickovic [96].

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**$\Delta_1$ -Coding.** Suppose  $V = L$  and the Condensation Condition holds for  $A$ . Then  $A$  is  $\Delta_1$  in a class-generic real  $R$ , preserving cardinals.

Now we are ready to begin the proof of the Theorem. Suppose  $\kappa$  is the least inaccessible and  $V = L$ . Let  $\langle \alpha_i | i < \kappa^+ \rangle$  be the increasing list of all  $\alpha \in (\kappa, \kappa^+)$  such that  $L_\alpha = \text{Skolem hull}(\kappa)$  in  $L_\alpha$ . For each  $i < \kappa^+$  define  $f_i : \kappa \rightarrow \kappa$  by  $f_i(\gamma) = \text{ordertype}(ORD \cap \text{Skolem hull}(\gamma) \text{ in } L_{\alpha_i})$ . By identifying  $f_i$  with its graph and using a pairing function we can think of  $f_i$  as a subset of  $\kappa$ . The following is straightforward.

**Lemma 1.** *Each  $f_i$  obeys the Condensation Condition. Indeed  $\langle f_i | i < \kappa^+ \rangle$  jointly obeys the Condensation Condition in the following sense: Suppose  $t$  is transitive,  $\delta$  is regular,  $\delta \in t$ ,  $x \in t$ . Then there exists a tight  $\delta$ -sequence  $x_0 \prec x_1 \prec \dots \prec t$  such that  $\text{card}(x_i) = \delta$ ,  $x \in x_0$  and each  $x_i$  strongly preserves all  $f_j$  for  $j \in x_i$  (and if  $\delta = \kappa$ , then we can alternatively require  $\text{card}(x_i) = \aleph_i$ ).*

Now, following Friedman [94] we use a “diagonally-supported” product of Jensen-style codings. For each  $i < \kappa^+$  let  $\mathcal{P}(i)$  be the forcing from Friedman-Velickovic [96] to make  $f_i$   $\Delta_1$ -definable in a class-generic real. Then  $\mathcal{P}$  consists of all  $p \in \prod_{i < \kappa^+} \mathcal{P}(i)$  such that for infinite cardinals  $\gamma$ ,  $\{i | p(i)(\gamma) \neq (\phi, \phi)\}$  has cardinality at most  $\gamma$  and in addition  $\{i | p(i)(0) \neq (\phi, \phi)\}$  is finite.

Now note that for successor cardinals  $\gamma < \kappa$  the forcing  $\mathcal{P}$  factors as  $\mathcal{P}_\gamma * \mathcal{P}^{G_\gamma}$  where  $\mathcal{P}_\gamma$  forces that  $\mathcal{P}^{G_\gamma}$  has the  $\gamma^+$ -CC. Also the joint Condensation Condition of Lemma 1 implies that the argument of Theorem 3 of Friedman-Velickovic [96] can be applied here to show that  $\mathcal{P}_\gamma$  is  $\leq \gamma$ -distributive, and also that  $\mathcal{P}$  is  $\Delta$ -distributive (if  $\langle D_i | i < \kappa \rangle$  is a sequence of predense sets, then it is dense to reduce each  $D_i$  below  $\aleph_{i+1}$ ). So  $\mathcal{P}$  preserves cofinalities.

Thus in a cardinal-preserving forcing extension of  $L$  we have produced  $\kappa^+$  reals  $\langle R_i | i < \kappa^+ \rangle$  where  $R_i$   $\Delta_1$ -codes  $f_i$  and hence there are wellorderings of  $\kappa$  of any length  $< \kappa^+$  which are  $\Delta_1$  in a real. Finally Lévy collapse to make  $\kappa = \omega_1$  and we have  $\delta^1_{\sim_2} = \omega_2$ ,  $\omega_1$  inaccessible to reals.  $\square$

The above proof also shows the following, which may be of independent interest.

**Theorem 2.** *Let  $\delta_{\sim_1}(\kappa)$  be the sup of the lengths of wellorderings of  $\kappa$  which are  $\Delta_{\sim_1}$  over  $L_\kappa[x]$  for some  $x$ , a bounded subset of  $\kappa$ . Then (relative to the consistency of an inaccessible) it is consistent that  $\kappa$  be weakly inaccessible and  $\delta_{\sim_1}(\kappa) = \kappa^+$ .*

*Remark.* The conclusion of Theorem 2 cannot hold in the context of sharps: if  $\kappa$  is weakly inaccessible and every bounded subset of  $\kappa$  has a sharp, then  $\delta_{\sim_1}(\kappa) < \kappa^+$ . This is because  $\delta_{\sim_1}(\kappa)$  is then the second uniform indiscernible for bounded subsets of  $\kappa$ , which can be written as the direct limit of the second uniform indiscernible for subsets of  $\delta$ , as  $\delta$  ranges over cardinals less than  $\kappa$ ; so  $\delta_{\sim_1}(\kappa)$  has cardinality  $\kappa$ .

Using the least inner model closed under sharp, we can also obtain the following.

**Theorem 3.** *Assuming it is consistent for every set to have a sharp, then this is also consistent with  $\delta^1_{\sim_3} = \omega_2$ .*

## REFERENCES

- Friedman [94], *A Large  $\Pi_2^1$  Set, Absolute for Set Forcings*, Proceedings of the American Mathematical Society, **122**, no. 1, 253–256. MR **95c**:03118
- Friedman-Velickovic [96],  *$\Delta_1$ -Definability* (to appear).
- Martin [77], *Descriptive Set Theory: Projective Sets*, Handbook of Mathematical Logic, Studies in Logic and the Foundations of Mathematics 90, Barwise (editor), pp. 783–815.
- Steel-Welch [?],  *$\Sigma_3^1$  Absoluteness and the Second Uniform Indiscernible*, Israel Journal of Mathematics (to appear).

DEPARTMENT OF MATHEMATICS, MASSACHUSETTS INSTITUTE OF TECHNOLOGY, CAMBRIDGE, MASSACHUSETTS 02139

*E-mail address:* `sdf@math.mit.edu`

*Current address:* Equipe de Logique, Université de Paris 7, 2, Place Jussieu, 75251 Paris Cedex 05, France

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF CALIFORNIA, BERKELEY, CALIFORNIA 94720

*E-mail address:* `woodin@math.berkeley.edu`