

## $p$ -SEQUENTIALITY AND $p$ -FRÉCHET-URYSOHN PROPERTY OF FRANKLIN COMPACT SPACES

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(Communicated by Franklin D. Tall)

ABSTRACT. Franklin compact spaces defined by maximal almost disjoint families of subsets of  $\omega$  are considered from the view of its  $p$ -sequentiality and  $p$ -Fréchet-Urysohn-property for ultrafilters  $p \in \omega^*$ . Our principal results are the following: CH implies that for every  $P$ -point  $p \in \omega^*$  there are a Franklin compact  $p$ -Fréchet-Urysohn space and a Franklin compact space which is not  $p$ -Fréchet-Urysohn; and, assuming CH, for every Franklin compact space there is a  $P$ -point  $q \in \omega^*$  such that it is not  $q$ -Fréchet-Urysohn. Some new problems are raised.

### INTRODUCTION

Let us recall some definitions. All spaces are assumed to be completely regular and Hausdorff. The Greek letter  $\omega$  will denote the first infinite ordinal. The set of all infinite subsets of  $\omega$  is denoted by  $[\omega]^\omega$ . The Stone-Čech compactification of  $\omega$  with the discrete topology is denoted by  $\beta(\omega)$ , which can be viewed as the set of all ultrafilters on  $\omega$ , where  $\widehat{A} = \text{cl}_{\beta(\omega)}(A) = \{q \in \beta(\omega) : A \in q\}$ , for  $A \subseteq \omega$ , is a basic clopen subset of  $\beta(\omega)$ . The remainder  $\omega^* = \beta(\omega) \setminus \omega$  is the set of all free ultrafilters on  $\omega$ . For  $A \in [\omega]^\omega$ , let  $A^*$  stand for  $\widehat{A} \setminus A$ . If  $f: X \rightarrow Y$  is a continuous function, then  $\overline{f}: \beta(X) \rightarrow \beta(Y)$  stands for the Stone-Čech extension of  $f$ . The *Rudin-Keisler* (pre-)order on  $\omega^*$  is defined by  $q \leq_{\text{RK}} p$  if there is a function  $f: \omega \rightarrow \omega$  such that  $\overline{f}(p) = q$ , for  $p, q \in \omega^*$ . The set of RK-predecessors of  $p \in \omega^*$  is denoted by  $P_{\text{RK}}(p) = \{q \in \omega^* : q \leq_{\text{RK}} p\}$ . We say that  $p$  is RK-equivalent to  $q$  and write  $p \approx_{\text{RK}} q$ , if  $p \leq_{\text{RK}} q$  and  $q \leq_{\text{RK}} p$ . The *Type* of  $p \in \omega^*$  is the set  $T(p) = \{q \in \omega^* : p \approx_{\text{RK}} q\}$  and it is not hard to see that  $p \approx_{\text{RK}} q$  iff there is a bijection  $f: \omega \rightarrow \omega$  such that  $\overline{f}(p) = q$ . If  $p \in \omega^*$ , then  $T(p)$  is a dense subset of  $\omega^*$ . A subset  $N \subseteq \omega^*$  is said to be *RK-dense* in  $\omega^*$  if for every  $p \in \omega^*$  there is  $q \in N$  such that  $q \leq_{\text{RK}} p$ . If  $\Sigma \subseteq [\omega]^\omega$ , then  $\Sigma^* = \{A^* : A \in \Sigma\}$ . An *almost disjoint* (AD) family of subsets of  $\omega$  is a family  $\Sigma$  of  $[\omega]^\omega$  such that if  $A, B \in \Sigma$  and  $A \neq B$ , then  $|A \cap B| < \omega$  or  $A^* \cap B^* = \emptyset$ . If  $\Sigma$  is not a proper subset of any AD family, then  $\Sigma$  is called a *maximal almost disjoint* (MAD) family.

Our fundamental concept is the following.

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Received by the editors July 5, 1993 and, in revised form, January 27, 1995.

1991 *Mathematics Subject Classification*. Primary 54A20, 54A35.

*Key words and phrases*. Ultrafilter, MAD family, Franklin compact space, Rudin-Keisler order,  $p$ -sequential,  $p$ -Fréchet Urysohn, ultra-sequential, ultra-Fréchet-Urysohn.

**1.1. Definition** (Katětov [Ka]). Let  $\mathcal{F}$  be a filter on  $\omega$ . A point  $x$  of a topological space  $X$  is a  $\mathcal{F}$ -limit point of the sequence  $(x_n)_{n < \omega}$  in  $X$ , we write  $x = \mathcal{F}\text{-lim } x_n$  if  $\{n < \omega : x_n \in V\} \in \mathcal{F}$  for every neighborhood  $V$  of  $x$ .

Let us note that if  $M$  is a nonempty closed subset of  $\omega^*$  and  $\mathcal{F}_M = \{A \in [\omega]^\omega : M \subseteq A^*\}$ , then  $M = \{p \in \omega^* : \mathcal{F}_M \subseteq p\} = \bigcap \{A^* : A \in \mathcal{F}_M\}$  and  $\mathcal{F}_M$  is a free filter on  $\omega$ . Conversely, if  $\mathcal{F}$  is a free filter on  $\omega$  and  $M_{\mathcal{F}} = \bigcap \{A^* : A \in \mathcal{F}\}$ , then  $\mathcal{F} = \{A \in [\omega]^\omega : M_{\mathcal{F}} \subseteq A^*\}$ ,  $M_{\mathcal{F}} = \{p \in \omega^* : \mathcal{F} \subseteq p\}$  and  $M_{\mathcal{F}}$  is a nonempty closed subset of  $\omega^*$ . Hence, we have that  $M = M_{\mathcal{F}_M}$  and  $\mathcal{F} = \mathcal{F}_{M_{\mathcal{F}}}$  for every nonempty closed subset  $M$  of  $\omega^*$  and for every free filter  $\mathcal{F}$  on  $\omega$ . Thus, throughout this paper, we will not distinguish between nonempty closed subsets of  $\omega^*$  and the corresponding free filters on  $\omega$ . This correspondence allows us to see that  $\mathcal{F}$ -limits, for an arbitrary free filter  $\mathcal{F}$  on  $\omega$ , may be defined in terms of  $p$ -limits for  $p \in \omega^*$  as is stated in the next lemma.

**1.2. Lemma.** *Let  $\mathcal{F}$  be a free filter on  $\omega$ , and let  $(x_n)_{n < \omega}$  be a sequence in a space  $X$ . Then,  $x = \mathcal{F}\text{-lim } x_n$  iff  $x = p\text{-lim } x_n$  for all  $p \in M_{\mathcal{F}}$ .*

*Proof.* Only the sufficiency requires proof. Assume that there is a neighborhood  $V$  of  $x$  such that  $A = \{n < \omega : x_n \in V\} \notin \mathcal{F}$ . Then,  $M_{\mathcal{F}}$  is not contained in  $A^*$ . Hence, we pick  $p \in M_{\mathcal{F}} \setminus A^*$ . But, we have that  $\{n < \omega : x_n \in V\} \in p$ , since  $x = p\text{-lim } x_n$ , which is a contradiction.

By using the  $\mathcal{F}$ -limits we may generalize sequentiality and Fréchet-Urysohn-property as follows:  $\square$

**1.3. Definition.** Let  $\mathcal{F}$  be a free filter on  $\omega$  and  $X$  a space. Then

- (1) (Kombarov [Ko])  $X$  is  $\mathcal{F}$ -sequential if for every nonclosed subset  $A$  of  $X$  there are a sequence  $(x_n)_{n < \omega}$  in  $A$  and  $x \in X \setminus A$  such that  $x = \mathcal{F}\text{-lim } x_n$ ;
- (2) (Comfort-Ponomarov-Savchenko)  $X$  is a  $\text{FU}(\mathcal{F})$ -space if for each  $A \subseteq X$  and each  $x \in \text{cl}(A)$  there is a sequence  $(x_n)_{n < \omega}$  in  $A$  such that  $x = \mathcal{F}\text{-lim } x_n$ ;
- (3) (Malykhin)  $X$  is ultra-sequential (resp., ultra-FU) if  $X$  is  $p$ -sequential (resp.,  $\text{FU}(p)$ -space) for every  $p \in \omega^*$ .

The next lemma is a direct consequence of Lemma 1.2.

**1.4. Lemma.** *Let  $\mathcal{F}$  be a free filter on  $\omega$  and  $X$  a space. Then*

- (1)  $X$  is  $\mathcal{F}$ -sequential iff for every nonclosed subset  $A$  of  $X$  there are a sequence  $(x_n)_{n < \omega}$  and  $x \in X \setminus A$  such that  $x = p\text{-lim } x_n$  for all  $p \in M_{\mathcal{F}}$ ;
- (2)  $X$  is a  $\text{FU}(\mathcal{F})$ -space iff for each  $A \subseteq X$  and each  $x \in \text{cl}(A)$  there is a sequence  $(x_n)_{n < \omega}$  in  $A$  such that  $x = p\text{-lim } x_n$  for all  $p \in M_{\mathcal{F}}$ .

Notice from 1.4 that if  $X$  is  $\mathcal{F}$ -sequential (resp., a  $\text{FU}(\mathcal{F})$ -space) for a free filter  $\mathcal{F}$  on  $\omega$ , then  $X$  is  $p$ -sequential (resp., a  $\text{FU}(p)$ -space) for all  $p \in M_{\mathcal{F}}$ . The converse does not hold in general; for instance, if  $q <_{\text{RK}} p$ , then the subspace  $\xi(q) = \omega \cup \{q\}$  of  $\beta(\omega)$  is both a  $\text{FU}(q)$ -space and a  $\text{FU}(p)$ -space, but it is not a  $\text{FU}(\mathcal{F}_M)$ -space, where  $M = \{p, q\}$ . If  $\mathcal{F}$  is a free filter on  $\omega$ , then  $\mathcal{F}$ -sequentiality and  $\text{FU}(\mathcal{F})$ -property coincide with the concepts of strong  $M_{\mathcal{F}}$ -sequentiality (see [Ko]) and  $\text{SFU}(M_{\mathcal{F}})$ -property (see [GT], [Koč]), respectively. If  $\mathcal{F}_\tau$  is the Fréchet filter on  $\omega$  (i.e., the filter generated by the cofinite subsets of  $\omega$ ), then sequentiality and Fréchet-Urysohn-property agree with  $\mathcal{F}_\tau$ -sequentiality and  $\text{FU}(\mathcal{F}_\tau)$ -property, respectively. More general, if  $\text{int}_{\omega^*}(M_{\mathcal{F}}) \neq \emptyset$  for a free filter  $\mathcal{F}$  on  $\omega$ , then sequentiality =  $\mathcal{F}$ -sequentiality and Fréchet-Urysohn-property =  $\text{FU}(\mathcal{F})$ -property. For a nonempty closed subset  $M$  of  $\omega^*$ ,  $\text{FU}(M)$ -space will mean  $\text{FU}(\mathcal{F}_M)$ -space.

The basic relationship between Rudin-Keisler order and the concepts given in Definition 1.3 is established in the following lemma.

**1.5. Lemma** (Ferreira [G]). *For  $p, q \in \omega^*$ , the following are equivalent.*

- (1)  $q \leq_{\text{RK}} p$ ;
- (2) every  $q$ -sequential space is  $p$ -sequential;
- (3) every  $\text{FU}(q)$ -space is a  $\text{FU}(p)$ -space.

For  $p \in \omega^*$  and a subset  $Y$  of a space  $X$ , we define  $Y^p = \{x \in X : x = p\text{-lim } x_n \text{ for some sequence } (x_n)_{n < \omega} \text{ in } Y\}$ . By induction, we define  $Y_0^p = Y$  and, for every ordinal number  $\nu$ ,  $Y_\nu^p = (Y_{\mu}^p)^p$  if  $\nu = \mu + 1$  and  $Y_\nu^p = \bigcup_{\lambda < \nu} Y_\lambda^p$  if  $\nu$  is a limit ordinal. Notice that if  $X$  is a  $p$ -sequential space, then  $\text{cl}_X(Y) = Y_{\omega_1}^p = \bigcup_{\nu < \omega_1} Y_\nu^p$  for each  $Y \subseteq X$ . Hence, we have that a space  $X$  is  $p$ -sequential iff the ordinal number

$$\sigma_p(X) = \min\{\nu \leq \omega_1 : \text{cl}_X(Y) = Y_\nu^p \text{ for all } Y \subseteq X\}$$

exists. If  $X$  is  $p$ -sequential, then  $\sigma_p(X)$  is called the *degree of  $p$ -sequentiality* of  $X$ . Observe that a space  $X$  is a  $\text{FU}(p)$ -space iff  $\sigma_p(X) = 1$ .

## 2. FRANKLIN COMPACT SPACES

We start with the definition of Franklin compact space: For a MAD family  $\Sigma$ , the Franklin compact space  $\mathcal{F}(\Sigma)$  of  $\Sigma$  is defined as follows:  $\mathcal{F}(\Sigma)$  is a factor-space of  $\beta(\omega)$  in which every  $A^*$ , for  $A \in \Sigma$ , is identified with a single point and  $\omega^* \setminus \bigcup \Sigma^*$  is also identified with a single point  $\infty$ ; hence, the point set of  $\mathcal{F}(\Sigma)$  is  $\omega \cup \{A^* : A \in \Sigma\} \cup \{\infty\}$ . Franklin compact spaces were introduced in [F], and they have been studied in many papers, for instance, [B<sub>1</sub>], [B<sub>2</sub>], [Ma<sub>1</sub>] and [Ma<sub>2</sub>]. It is known that Franklin compact spaces are sequential spaces with degree of sequentiality equal to 2 and are not Fréchet-Urysohn. In this section, we will consider these spaces from the view of  $p$ -sequentiality and  $\text{FU}(p)$ -property for  $p \in \omega^*$ . So  $\mathcal{F}(\Sigma)$  is ultra-sequential and its degree of  $p$ -sequentiality is not greater than 2, for every  $p \in \omega^*$ . The next two lemmas are fundamental in the study of the  $p$ -sequentiality and the  $\text{FU}(p)$ -property of Franklin compact spaces.

**2.1. Lemma.** *Let  $p \in \omega^*$  and  $\Sigma$  a MAD family. Then,  $\mathcal{F}(\Sigma)$  is not a  $\text{FU}(p)$ -space if and only if there is  $C \in [\omega]^\omega$  such that*

$$C^* \setminus \bigcup \Sigma^* \neq \emptyset \quad \text{and} \quad P_{\text{RK}}(p) \cap C^* \subseteq \bigcup \Sigma^*.$$

*Proof. Necessity:* If  $\sigma_p(\mathcal{F}(\Sigma)) = 2$ , then there must be  $C \in [\omega]^\omega$  such that  $\infty \in \text{cl}(C)$  and no sequence of  $C$   $p$ -converges to  $\infty$ . Let  $q \in P_{\text{RK}}(p) \cap C^*$ . Then, we may find  $f: \omega \rightarrow C$  such that  $\bar{f}(p) = q$ . Since  $(f(n))_{n < \omega}$  is a sequence in  $C$  and  $q = p\text{-lim } f(n)$ , we have that  $q \in A^*$  for some  $A \in \Sigma$ . This shows that  $P_{\text{RK}}(p) \cap C^* \subseteq \bigcup \Sigma^*$ .

*Sufficiency:* Let  $C \in [\omega]^\omega$  satisfy that  $C^* \setminus \bigcup \Sigma^* \neq \emptyset$  and  $P_{\text{RK}}(p) \cap C^* \subseteq \bigcup \Sigma^*$ . It is enough to prove that no sequence of  $C$   $p$ -converges to  $\infty$  in  $\mathcal{F}(\Sigma)$ . Indeed, let  $f: \omega \rightarrow C$  be such that  $\bar{f}(p) = q \in \omega^*$ . Then  $q \in P_{\text{RK}}(p) \cap C^* \subseteq \bigcup \Sigma^*$  and so there is  $A \in \Sigma$  for which  $q \in A^*$ ; hence,  $A^* = p\text{-lim } f(n)$ .  $\square$

**2.2. Lemma** (Boldjiev and Malykhin [BM]). *Let  $\Sigma$  be an AD family. If  $p \in \omega^*$  is not a  $P$ -point, then  $T(p)$  is not contained in  $\bigcup \Sigma^*$ .*

From 2.2 it follows that if  $T(p) \subseteq \bigcup \Sigma^*$  for some AD family  $\Sigma$ , then  $p$  must be a  $P$ -point. The following theorem is proved in [BM]: now it can be considered as a consequence of Lemmas 2.1 and 2.2.

**2.3. Theorem** (Boldjiev and Malykhin). *Every Franklin compact space is a FU( $p$ )-space for each non- $P$ -point  $p \in \omega^*$ .*

**2.4. Corollary** (Boldjiev and Malykhin). *If there are not  $P$ -points in  $\omega^*$ , then every Franklin compact space is ultra-FU.*

The existence of  $P$ -points in  $\omega^*$  is independent of the axioms of ZFC. In fact, W. Rudin [R] showed that CH implies the existence of  $P$ -points in  $\omega^*$  and Shelah [W], [Mi] constructed a model of ZFC in which there are not  $P$ -points in  $\omega^*$ . Hence, in Shelah's model the Franklin compact spaces are ultra-FU. So, it is natural to ask whether every Franklin compact space is not a FU( $p$ )-space whenever  $p \in \omega^*$  is a  $P$ -point. We shall show that CH implies that for every  $P$ -point  $p \in \omega^*$  there is a MAD family  $\Sigma$  such that  $\mathcal{F}(\Sigma)$  is not a FU( $p$ )-space.

Let  $\rho$  be the smallest cardinal, such that if  $\mathcal{A}$  is a centered family of clopen subsets of  $\omega^*$  and  $|\mathcal{A}| < \mathfrak{c}$ , then  $\text{int}(\bigcap \mathcal{A}) \neq \emptyset$ . It is well known that  $\rho = \mathfrak{c}$  under MA.

**2.5. Theorem** ( $\rho = \mathfrak{c}$ ). *If  $N$  is a nowhere dense closed subset of  $\omega^*$ , then there is a MAD family  $\Sigma$  such that  $N \subseteq \omega^* \setminus \bigcup \Sigma^*$  and  $\mathcal{F}(\Sigma)$  is a FU( $N$ )-space; hence,  $\mathcal{F}(\Sigma)$  is a FU( $p$ )-space for all  $p \in N$ .*

*Proof.* The desired family will be constructed by transfinite induction. Enumerate the set  $\{C \in [\omega]^\omega : C^* \cap N = \emptyset\}$  as  $\{C_\alpha : \alpha < \mathfrak{c}\}$ . We define  $A_0 = C_0$  and  $N_0 = N$ . Suppose that  $A_\beta \in [\omega]^\omega$  and a nowhere dense closed subset  $N_\beta$  have been defined for each  $\beta < \alpha < \mathfrak{c}$ , where  $\alpha$  is a fixed ordinal smaller than  $\mathfrak{c}$ , so that

- (1)  $\Sigma_\beta = \{A_\gamma : \gamma < \beta\}$  is AD for each  $\beta < \alpha$ ;
- (2)  $(\bigcup \Sigma_\beta^*) \cap (\bigcup_{\gamma < \beta} N_\gamma) = \emptyset$  for each  $\beta < \alpha$ ;
- (3)  $N_\beta \subseteq \bigcup \{P_{\text{RK}}(p) : p \in N\}$  and  $N_\beta \cap P_{\text{RK}}(p) \neq \emptyset$  for each  $\beta < \alpha$ ; and
- (4) if  $C_\beta^*$  is not contained in  $\bigcup \Sigma_\beta^*$ , then  $A_\beta^* \subseteq C_\beta^* \setminus [(\bigcup \Sigma_\beta^*) \cup (\bigcup_{\gamma \leq \beta} N_\gamma)]$ .

Consider  $C_\alpha^*$  and  $\Sigma_\alpha = \{A_\beta : \beta < \alpha\}$ . If  $C_\alpha^* \subseteq \bigcup \Sigma_\alpha^*$ , then we let  $N_\alpha = \emptyset$ . If this is not the case, then put  $W = \text{int}(C_\alpha^* \setminus [(\bigcup \Sigma_\alpha^*) \cup (\bigcup_{\beta < \alpha} N_\beta)])$ . Since  $\rho = \mathfrak{c}$ , we have that  $W \neq \emptyset$ . Hence, we choose  $B \in [\omega]^\omega$  so that  $B^* \subseteq W$ . Now, we fix a one-to-one function  $f_\alpha : \omega \rightarrow B$  and set  $N_\alpha = \overline{f_\alpha(N)}$ . Then we choose  $A_\alpha \in [\omega]^\omega$  such that  $A_\alpha^* \subseteq B^* \setminus N_\alpha$ . Thus, we have defined  $A_\alpha$  and  $N_\alpha$ . It is not hard to verify that  $\Sigma = \{A_\alpha : \alpha < \mathfrak{c}\}$  is the required MAD family.

It follows from 2.5, in particular, that if  $\rho = \mathfrak{c}$  and  $K \subseteq \omega^*$  satisfies  $|K| < \mathfrak{c}$ , then there is a MAD family  $\Sigma$  such that  $\mathcal{F}(\Sigma)$  is a FU( $p$ )-space for every  $p \in \text{cl}(K)$ .  $\square$

The next result, for only one  $P$ -point of  $\omega^*$ , is contained in [BM, Example 1], but there is a small defect in its proof there. Here, we give a more general statement and the correct proof.

**2.6. Theorem** (CH). *If  $p_\alpha \in \omega^*$  is a  $P$ -point for every  $\alpha < \mathfrak{c}$ , then there exists a MAD family  $\Sigma$  such that  $\bigcup_{\alpha < \mathfrak{c}} P_{\text{RK}}(p_\alpha) \subseteq \bigcup \Sigma^*$  and hence  $\sigma_{p_\alpha}(\mathcal{F}(\Sigma)) = 2$  for each  $\alpha < \mathfrak{c}$ .*

*Proof.* Let  $p_\alpha \in \omega^*$  be a  $P$ -point for every  $\alpha < \mathfrak{c}$ . Notice that each element of  $\bigcup_{\alpha < \mathfrak{c}} P_{\text{RK}}(p_\alpha)$  is also a  $P$ -point of  $\omega^*$ . We may enumerate  $\bigcup_{\alpha < \mathfrak{c}} P_{\text{RK}}(p_\alpha)$  as  $\{q_\mu : \mu < \omega_1\}$ ; this is possible because  $|\bigcup_{\alpha < \mathfrak{c}} P_{\text{RK}}(p_\alpha)| = 2^\omega = \omega_1$ . To define the MAD family we proceed by transfinite induction. Let  $\{A_n : n < \omega\}$  be a partition of  $\omega$  in infinite subsets. Suppose that  $A_\mu$  has been defined for  $\mu < \nu < \omega_1$  so that

- (1)  $\{A_\mu : \mu < \nu\}$  is an AD family; and

$$(2) \{q_\mu : \mu < \nu\} \subseteq \bigcup \{A_\mu^* : \mu < \nu\}.$$

If  $q_\nu \in \bigcup \{A_\mu^* : \mu < \nu\}$ , then we choose any  $A_\nu \in [\omega]^\omega$  such that  $|A_\nu \cap A_\mu| < \omega$  for all  $\mu < \nu$ . Otherwise, we take  $A_\nu \in q_\nu$  so that  $A_\nu^* \cap A_\mu^* = \emptyset$  for every  $\mu < \nu$ . Set  $\Sigma = \{A_\mu : \mu < \nu\}$ . By construction, we have that  $\bigcup_{\alpha < \mathfrak{c}} P_{\text{RK}}(p_\alpha) \subseteq \bigcup \{A_\mu^* : \mu < \omega_1\}$ . Now, if  $A \in [\omega]^\omega$ , then there are  $\mu, \nu < \omega_1$  such that  $q_\mu \in A^* \cap A_\nu^*$  and so  $A \cap A_\nu$  is infinite. This shows that  $\Sigma$  is a MAD family. From 2.1 it follows that  $\sigma_{p_\alpha}(\mathcal{F}(\Sigma)) = 2$  for each  $\alpha < \mathfrak{c}$ .

A solution to the following problem could provide some information about the FU( $p$ )-property of Franklin compact spaces for the case when  $p \in \omega^*$  is a  $P$ -point.  $\square$

**2.7. Problem.** Is it possible that an ultra-FU Franklin compact space exists if  $P$ -points of  $\omega^*$  exist?

The proof of the next theorem is analogous to the previous one.

**2.8. Theorem (MA).** For every  $P_\mathfrak{c}$ -point  $p \in \omega^*$  there exists a MAD family  $\Sigma$  such that  $P_{\text{RK}}(p) \subseteq \bigcup \Sigma^*$  and hence  $\sigma_p(\mathcal{F}(\Sigma)) = 2$ .

**2.9. Problem.** Assume  $[MA + \neg CH]$  and let  $p \in \omega^*$  be a  $P_\kappa$ -point for some  $\omega < \kappa < \mathfrak{c}$  (such  $P_\kappa$ -point exists under  $[MA + \neg CH]$ , see [S]).

Is there a MAD family  $\Sigma$  family that:

- (a)  $T(p) \subseteq \bigcup \Sigma^*$  or
- (b)  $P_{\text{RK}}(p) \subseteq \bigcup \Sigma^*$ ?

We turn now to prove that CH implies that for every MAD family  $\Sigma$  there is a  $P$ -point  $q \in \omega^*$  such that  $\mathcal{F}(\Sigma)$  is not a FU( $q$ )-space. We need the next lemma.

**2.10. Lemma.** Let  $\Sigma$  be a MAD family. If  $f: \omega \rightarrow \omega$  is finite-to-one and onto, then  $f^{-1}[\Sigma] = \{f^{-1}(A) : A \in \Sigma\}$  is also a MAD family.

*Proof.* Let  $T = f^{-1}[\Sigma]$ . Since  $\Sigma$  is AD and  $f$  is finite-to-one, we have that  $|f^{-1}(A) \cap f^{-1}(B)| < \omega$  whenever  $A, B \in \Sigma$  and  $A \neq B$ . Now, let  $B \in [\omega]^\omega$ . Since  $f[B] \in [\omega]^\omega$ , there is  $A \in \Sigma$  such that  $|A \cap f[B]| = \omega$  and so  $|f^{-1}(A) \cap B| = \omega$ . Thus,  $T$  is a MAD family.  $\square$

**2.11. Theorem (CH).** For every MAD family  $\Sigma$  there is a  $P$ -point  $p \in \omega^*$  such that  $P_{\text{RK}}(p) \subseteq \bigcup \Sigma^*$  and hence  $\sigma_p(\mathcal{F}(\Sigma)) = 2$ .

*Proof.* Let  $\mathcal{F} = \{f \mid f: \omega \rightarrow \omega \text{ is finite-to-one and onto}\}$  and  $\Sigma = \{A_\alpha : \alpha < \omega_1\}$ . Enumerate  $\mathcal{F}$  as  $\{f_\alpha : \alpha < \omega_1\}$ . We proceed by transfinite induction. Set  $B_0 = A_0$ , and assume that  $B_\beta$  and  $A_{\gamma_\beta} \in \Sigma$  have been defined for each  $\beta < \alpha$  so that

$$(1) B_\beta^* \subseteq [B_0 \cap f_\beta^{-1}(A_{\gamma_\beta})]^* \text{ for } \beta < \alpha; \text{ and}$$

(2)  $\{B_\beta : \beta < \alpha\}$  has the finite intersection property. Since  $\{B_\beta : \beta < \alpha\}$  has the finite intersection property, there is  $D \in [\omega]^\omega$  for which  $D^* \subseteq \bigcap_{\beta < \alpha} B_\beta^*$ . By Lemma 2.10, we may find  $A_{\gamma_\alpha} \in \Sigma$  such that  $|D \cap f_\alpha^{-1}(A_{\gamma_\alpha})| = \omega$ . Then, we put  $B_\alpha = D \cap f_\alpha^{-1}(A_{\gamma_\alpha})$ . Thus,  $\{B_\alpha : \alpha < \omega_1\}$  has the finite intersection property and hence there is  $p \in \omega^*$  such that  $\{B_\alpha : \alpha < \omega_1\} \subseteq p$ . So  $p \in A_0^*$ . If  $f: \omega \rightarrow \omega$  is a bijection, then  $f = f_\alpha$  for some  $\alpha < \omega_1$ ; hence,  $\bar{f}(p) \in A_{\gamma_\alpha}^*$ . This shows that  $T(p) \subseteq \bigcup \Sigma^*$ . According to 2.2, we have that  $p$  must be a  $P$ -point of  $\omega^*$ . Notice, in general, that if  $q \leq_{\text{RK}} p$  and  $\underline{p} \in \omega^*$  is a  $P$ -point, then there is a finite-to-one surjection  $f: \omega \rightarrow \omega$  such that  $\bar{f}(p) = q$ . We claim that  $P_{\text{RK}}(p) \subseteq \bigcup \Sigma^*$ . Indeed, fix  $q \in P_{\text{RK}}(p)$ . Then, there is  $\alpha < \omega_1$  such that  $\bar{f}_\alpha(p) = q$ . Since  $p \in [f_\alpha^{-1}(A_{\gamma_\alpha})]^*$ ,  $q = \bar{f}_\alpha(p) \in A_{\gamma_\alpha}^*$ . This proves the claim.  $\square$

In a similar way we may prove the following.

**2.12. Theorem (MA).** *For every MAD family  $\Sigma$  there is a  $P_c$ -point  $p \in \omega^*$  such that  $P_{\text{RK}}(p) \subseteq \bigcup \Sigma^*$  and hence  $\sigma_p(\mathcal{F}(\Sigma)) = 2$ .*

We could predict something like Theorems 2.14 and 2.15 after knowing Malykhin's result [Ma<sub>3</sub>, Theorem 1], about the coincidence of ultra-FU and ultra-sequentiality with Fréchet-Urysohn property and sequentiality, respectively, under the assumption  $n(\omega^*) > \mathfrak{c}$ , where  $n(X)$  is the Novak number of  $X$  (i.e., the smallest power of a family of nowhere dense subsets of  $X$  covering  $X$ ). Indeed, under CH, MA or  $n(\omega^*) > \mathfrak{c}$  a Franklin compact space  $\mathcal{F}(\Sigma)$  cannot be ultra-FU (i.e., there is  $p \in \omega^*$  such that  $\sigma_p(\mathcal{F}(\Sigma)) > 1$ ) since it is not Fréchet-Urysohn. As a consequence of this result and Lemma 1.5 we have the following two theorems.

**2.13. Problem.** Under CH, is it true that for every MAD family  $\Sigma$  there is a  $P$ -point  $p \in \omega^*$  such that  $\mathcal{F}(\Sigma)$  is a  $\text{FU}(p)$ -space?

**2.14. Theorem.** *Let  $\Sigma$  be a MAD family. Then,  $\mathcal{F}(\Sigma)$  is ultra-FU if and only if  $\{p \in \omega^* : \mathcal{F}(\Sigma) \text{ is a } \text{FU}(p)\text{-space}\}$  is RK-dense in  $\omega^*$ .*

The next result is a direct application of Malykhin's Theorem quoted above and 2.14.

**2.15. Theorem ( $n(\omega^*) > \mathfrak{c}$ ).** *Let  $\Sigma$  be a MAD family. Then, the set  $\{p \in \omega^* : \mathcal{F}(\Sigma) \text{ is a } \text{FU}(p)\text{-space}\}$  is not RK-dense in  $\omega^*$ .*

We formulate and prove the following proposition that is stronger than Malykhin's theorem.

**2.16. Proposition ( $n(\omega^*) > \mathfrak{c}$ ).** *Let  $\mathfrak{C}$  be a family of nonempty closed subsets of  $\omega^*$  such that  $|\mathfrak{C}| \leq \mathfrak{c}$  and  $\bigcup \mathfrak{C}$  is RK-dense in  $\omega^*$ . Then,  $\text{int}(F) \neq \emptyset$  for some  $F \in \mathfrak{C}$ .*

*Proof.* We have that for every  $p \in \omega^*$  there are  $F_p \in \mathfrak{C}$  and a function  $f_p: \omega \rightarrow \omega$  such that  $p \in \overline{f_p^{-1}(F_p)}$ . Hence,  $\{\overline{f_p^{-1}(F_p)} : p \in \omega^*\}$  is a family of nonempty closed subsets of  $\omega^*$  and covers  $\omega^*$ . But this family has power less than or equal to  $\mathfrak{c}$ . So there must be  $F \in \mathfrak{C}$  such that  $\text{int}(F) \neq \emptyset$ .

We proceed to prove, under the assumption of  $n(\omega^*) > \mathfrak{c}$ , a result more general than the one given in Theorem 2.11: it is an immediate consequence of 2.1 and 2.16.  $\square$

**2.17. Theorem ( $n(\omega^*) > \mathfrak{c}$ ).** *Let  $\Sigma_\alpha$  be a MAD family for every  $\alpha < \mathfrak{c}$ . Then, there is (a  $P$ -point)  $p \in \omega^*$  such that  $\mathcal{F}(\Sigma_\alpha)$  is not a  $\text{FU}(p)$ -space for all  $\alpha < \mathfrak{c}$ .*

*Proof.* For each  $\alpha < \mathfrak{c}$ , put  $F_\alpha = \omega^* \setminus \bigcup \Sigma_\alpha^*$ . Consider the set  $\mathfrak{C} = \{F_\alpha : \alpha < \mathfrak{c}\}$ . Since  $\text{int}(F_\alpha) = \emptyset_\alpha$  for each  $\alpha < \mathfrak{c}$ , by 2.16, there is  $p \in \omega^*$  such that  $P_{\text{RK}}(p) \cap \bigcup \mathfrak{C} = \emptyset$  and so  $P_{\text{RK}}(p) \subseteq \omega^* \setminus \bigcup \mathfrak{C} = \bigcap_{\alpha < \mathfrak{c}} [\bigcup \Sigma_\alpha^*]$ . Lemma 2.1 implies that  $\mathcal{F}(\Sigma_\alpha)$  is not a  $\text{FU}(p)$ -space for all  $\alpha < \mathfrak{c}$ .  $\square$

The next problem now appears to be natural.

**2.18. Problem.** Is it possible that some  $p \in \omega^*$  exists such that every Franklin compact space is not a  $\text{FU}(p)$ -space?

Perhaps it will be so in models obtained from GCH and by adding greater than  $\omega_1$  of new Cohen reals.

We recall that, for  $p \in \omega^*$ , a space  $X$  is a *strictly*  $FU(p)$ -space if  $x \in \bigcap_{n < \omega} \text{cl}_X(Y_n)$ , where  $Y_n \subseteq X$  for each  $n < \omega$ ; then for each  $n < \omega$  there is  $x_n \in Y_n$  such that  $x = p\text{-lim } x_n$ .

**2.19. Problem.** Let  $p \in \omega^*$ , and let  $\Sigma$  be a MAD family. Can  $\mathcal{F}(\Sigma)$  be a strictly  $FU(p)$ -space?

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