

## DUALITY AND PERFECT PROBABILITY SPACES

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ABSTRACT. Given probability spaces  $(X_i, \mathcal{A}_i, P_i)$ ,  $i = 1, 2$ , let  $\mathcal{M}(P_1, P_2)$  denote the set of all probabilities on the product space with marginals  $P_1$  and  $P_2$  and let  $h$  be a measurable function on  $(X_1 \times X_2, \mathcal{A}_1 \otimes \mathcal{A}_2)$ . Continuous versions of linear programming stemming from the works of Monge (1781) and Kantorovich-Rubinstein (1958) for the case of compact metric spaces are concerned with the validity of the duality

$$\begin{aligned} & \sup\left\{\int h \, dP : P \in \mathcal{M}(P_1, P_2)\right\} \\ &= \inf\left\{\sum_{i=1}^2 \int h_i \, dP_i : h_i \in \mathcal{L}^1(P_i) \text{ and } h \leq \oplus_i h_i\right\} \end{aligned}$$

(where  $\mathcal{M}(P_1, P_2)$  is the collection of all probability measures on  $(X_1 \times X_2, \mathcal{A}_1 \otimes \mathcal{A}_2)$  with  $P_1$  and  $P_2$  as the marginals). A recently established general duality theorem asserts the validity of the above duality whenever at least one of the marginals is a perfect probability space. We pursue the converse direction to examine the interplay between the notions of duality and perfectness and obtain a new characterization of perfect probability spaces.

### 1. INTRODUCTION

Let  $(X, \mathcal{A}, P)$  be a probability space.  $P$  is called *perfect* (equivalently, the space  $(X, \mathcal{A}, P)$  is called *perfect*) if, for every  $\mathcal{A}$ -measurable, real-valued function  $f$  on  $X$  we can find a Borel subset  $B_f$  of the real line such that  $B_f \subset f(X)$  with  $P(f^{-1}(B_f)) = 1$ . Introduced by Gnedenko and Kolmogorov [2] “to achieve complete harmony between the abstract theory of measure and the theory of measures in metric spaces”, perfect probability spaces form the technically most pleasing class of probability spaces (see Ramachandran [10]).

A succinct history of the notion of duality from its origins in the works of Monge [5] and Kantorovich-Rubinstein [3], along with a variety of applications in probability theory can be found in Kellerer [4] (see also Dudley [1] and Rachev [9]). Ramachandran and Rüschendorf [11] have recently established that a general duality theorem holds whenever one of the underlying spaces is perfect.

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In this note we investigate the converse direction and obtain a new characterization of perfect probability spaces using the notion of duality which brings to light the interplay between the notions of duality and perfectness.

## 2. NOTATION AND PRELIMINARIES

We use customary measure-theoretic terminology and notation (as, for instance, in Neveu [6]). All measures that we consider are probabilities. For properties of perfect measures we refer the reader to Ramachandran [10]. We denote by  $\mathcal{B}$  the  $\sigma$ -algebra of Borel subsets of  $[0, 1]$  and by  $\lambda$  the Lebesgue measure on  $([0, 1], \mathcal{B})$ .  $\Delta$  denotes the diagonal in  $[0, 1] \times [0, 1]$ . If  $P$  is a probability on  $(X, \mathcal{A})$ , then  $P_*$  and  $P^*$  denote respectively the inner and the outer measures induced by  $P$ . A  $\sigma$ -algebra  $\mathcal{A}_0$  is said to be countably generated (or c.g. for short) if  $\mathcal{A}_0 = \sigma(\{A_n, n \geq 1\})$ , in which case  $\varphi : (X, \mathcal{A}_0) \rightarrow ([0, 1], \mathcal{B})$  defined by  $\varphi(x) = \sum_{n=1}^{\infty} (2/3^n) 1_{A_n}(x)$  is called the Marczewski function;  $\varphi$  is measurable with  $\varphi(x_1) \neq \varphi(x_2)$  if  $x_1$  and  $x_2$  belong to different atoms of  $\mathcal{A}_0$ , and so we can identify  $(X, \mathcal{A}_0)$  with  $(\varphi(X), \mathcal{B} \cap \varphi(X))$ . We say that  $(X, \mathcal{A}, P)$  is a thick subspace of  $(X_1, \mathcal{A}_1, P_1)$  and write  $(X, \mathcal{A}, P) \subset (X_1, \mathcal{A}_1, P_1)$  whenever  $X \subset X_1$ ,  $\mathcal{A} = \mathcal{A}_1 \cap X =$  the trace of  $\mathcal{A}_1$  on  $X$ ,  $P_1^*(X) = 1$  and  $P = P_1^*|_{\mathcal{A}}$ .

Let  $(X_i, \mathcal{A}_i, P_i)$ ,  $i = 1, 2$ , be two probability spaces. A probability  $\mu$  on  $(X_1 \times X_2, \mathcal{A}_1 \otimes \mathcal{A}_2)$  is said to have marginals  $P_1$  and  $P_2$  if

$$\begin{aligned} \mu(A_1 \times X_2) &= P_1(A_1) \text{ for all } A_1 \in \mathcal{A}_1, \text{ and} \\ \mu(X_1 \times A_2) &= P_2(A_2) \text{ for all } A_2 \in \mathcal{A}_2. \end{aligned}$$

Let  $\mathcal{M}(P_1, P_2) = \{ \mu \text{ on } \mathcal{A}_1 \otimes \mathcal{A}_2 : \mu \text{ has marginals } P_1 \text{ and } P_2 \}$ .  $\pi_i : X_1 \times X_2 \rightarrow X_i$  denote the canonical projections for  $i = 1, 2$ . The abbreviation  $\oplus g_i$  is used for  $\sum_{i=1}^2 g_i \circ \pi_i$ .

For a bounded,  $\mathcal{A}_1 \otimes \mathcal{A}_2$ -measurable function  $h$  on  $X_1 \times X_2$ , the marginal problem is concerned with

$$S(h) = \sup \left\{ \int_{X_1 \times X_2} h d\mu : \mu \in \mathcal{M}(P_1, P_2) \right\}$$

while the dual problem deals with

$$I(h) = \inf \left\{ \sum_{i=1}^2 \int_{X_i} h_i dP_i : h_i \in \mathcal{L}^1(P_i) \text{ and } h \leq \oplus_i h_i \right\}.$$

The measure-theoretic version of the transportation problem dating back to Monge [5] seeks the validity of the duality

$$(D) \quad S(h) = I(h).$$

The main duality theorem of Kellerer [4] deals essentially with second countable or metrizable spaces  $X_i$ ,  $i = 1, 2$ , with tight (or Radon) probabilities defined on the Borel sets in which case (D) is shown to hold for a suitably large class containing all the bounded, measurable functions. The following result of Ramachandran and Rüschemdorf [11] is the most general duality theorem of this type.

**Theorem 1.** *If at least one of the underlying probability spaces is perfect, then (D) holds for all bounded, measurable functions.*

We pursue the converse direction in the next section.

3. MAIN RESULTS

We first construct an example to show that the direct converse of Theorem 1 does not hold; that is, (D) can hold for all bounded measurable functions while both the underlying probability spaces are nonperfect.

**Example 1.** Pachl [8] has constructed a subset  $X$  of  $[0, 1]$  such that

- (i)  $\lambda^*(X) = 1 = \lambda^*([0, 1] - X)$  and
- (ii) if  $\bar{\mu}$  is a probability on  $([0, 1] \times [0, 1], \mathcal{B} \otimes \mathcal{B})$  with  $\bar{\mu} \in \mathcal{M}(\lambda, \lambda)$ , then  $\bar{\mu}^*(X \times X) = 1$ .

Let  $X_i = X$ ,  $\mathcal{A}_i = \mathcal{B} \cap X$  and  $P_i = \lambda^*|_{\mathcal{A}_i}$  for  $i = 1, 2$ . Then  $P_i$  is clearly nonperfect for both  $i = 1, 2$ . For  $\mu \in \mathcal{M}(P_1, P_2)$  let  $\bar{\mu}$  be defined by  $\bar{\mu}(C) = \mu(C \cap (X \times X))$ ,  $C \in \mathcal{B} \otimes \mathcal{B}$ . Then it is easy to check that  $\bar{\mu} \in \mathcal{M}(\lambda, \lambda)$  and that  $\mu \rightarrow \bar{\mu}$  is a 1-1 correspondence between  $\mathcal{M}(P_1, P_2)$  and  $\mathcal{M}(\lambda, \lambda)$ . Further, since  $\mathcal{A}_1 \otimes \mathcal{A}_2 = \mathcal{B} \otimes \mathcal{B} \cap (X \times X)$ , it can be checked (starting with  $1_C$ ,  $C \in \mathcal{A}_1 \otimes \mathcal{A}_2$  and using standard measure-theoretic arguments) that for every  $h$  on  $X_1 \times X_2$  which is  $\mathcal{A}_1 \otimes \mathcal{A}_2$ -measurable there is  $\bar{h}$  on  $[0, 1] \times [0, 1]$  which is  $\mathcal{B} \otimes \mathcal{B}$ -measurable such that  $\bar{h}|_{X_1 \times X_2} = h$  and  $\int h d\mu = \int \bar{h} d\bar{\mu}$ . Hence it follows that for every bounded,  $\mathcal{A}_1 \otimes \mathcal{A}_2$ -measurable  $h$ ,

$$\begin{aligned} S(h) &= \sup\{\int h d\mu : \mu \in \mathcal{M}(P_1, P_2)\} \\ &= \sup\{\int \bar{h} d\bar{\mu} : \bar{\mu} \in \mathcal{M}(\lambda, \lambda)\} \\ &= S(\bar{h}) \\ &= I(\bar{h}) \text{ by Theorem 1} \\ &\geq I(h) \text{ (by the definition of } I(h)) \\ &\geq S(h) \text{ (by the definitions of } I(h) \text{ and } S(h)) \end{aligned}$$

whereby (D) holds.

In order to bring out the interplay between the notions of perfectness and duality we introduce

**Definition 1.** A probability space  $(X_1, \mathcal{A}_1, P_1)$  is said to be a duality space if for every  $(X_2, \mathcal{A}_2, P_2)$  the duality (D) holds for all bounded, measurable functions.

Theorem 1 can now be recast as

**Theorem 2.** *Every perfect probability space is a duality space.*

The next major step is to establish

**Proposition 1.** *If  $(X, \mathcal{A}, P)$  is nonperfect, then there exists  $(X_1, \mathcal{A}_1, P_1)$  such that*

- (i)  $(X, \mathcal{A}, P) \subset (X_1, \mathcal{A}_1, P_1)$  and
- (ii)  $(X_1, \mathcal{A}_1, P_1)$  is not a duality space.

*Proof.* Since  $(X, \mathcal{A}, P)$  is nonperfect, there exists a c.g. sub- $\sigma$ -algebra  $\mathcal{A}_0$  of  $\mathcal{A}$  such that  $P_0 = P|_{\mathcal{A}_0}$  is nonperfect (see P3 of Ramachandran [10, p. 26]). Let  $\varphi$  be the Marczewski function on  $(X, \mathcal{A}_0)$  and let  $Q_0 = P_0\varphi^{-1}$ . Then  $\varphi$  provides the Marczewski identification of the space  $(X, \mathcal{A}_0, P_0)$  with  $(\varphi(X), \mathcal{B} \cap \varphi(X), Q_0) \subset ([0, 1], \mathcal{B}, \bar{Q}_0)$  where  $\bar{Q}_0$  is defined on  $\mathcal{B}$  by  $\bar{Q}_0(B) = Q_0(B \cap \varphi(X)) (= P_0\varphi^{-1}(B))$ ,  $B \in \mathcal{B}$ . Notice that, by construction,  $\bar{Q}_0^*(\varphi(X)) = 1$ .

Suppose that  $\bar{Q}_{0*}(\varphi(X)) = 1$ . Then we can find a set  $Y \subset \varphi(X)$ ,  $Y \in \mathcal{B}$ , with  $\bar{Q}_0(Y) = 1$ . Since  $Y$  is a Borel subset of  $[0, 1]$ , every probability on  $(Y, \mathcal{B} \cap Y)$  is perfect (see Theorem 2.3.1 of Ramachandran [10]); hence,  $Q_0|_Y$  is perfect which in turn would imply that  $Q_0$  is perfect. However,  $Q_0$  is nonperfect since

$P_0$  is nonperfect (see P7 of Ramachandran [10, p. 27]), and so we conclude that  $\overline{Q}_{0*}(\varphi(X)) = \alpha < 1$ .

Let  $f : x \mapsto (x, \varphi(x), \varphi(x)), x \in X$ .

Then  $f: (X, \mathcal{A}, P) \rightarrow (X \times \varphi(X) \times [0, 1], \mathcal{A} \otimes (\mathcal{B} \cap \varphi(X)) \otimes \mathcal{B}, Pf^{-1})$  has the following properties:

1.  $f$  is 1-1.
2. Since  $G_\varphi = \{(x, \varphi(x)), x \in X\} = \text{Graph of } \varphi \in \mathcal{A} \otimes \mathcal{B} \cap (X \times \varphi(X)) = \mathcal{A} \otimes (\mathcal{B} \cap \varphi(X))$ , it follows that

$$f(X) = (G_\varphi \times [0, 1]) \cap (X \times (\Delta \cap (\varphi(X) \times [0, 1]))) \in \mathcal{A} \otimes (\mathcal{B} \cap \varphi(X)) \otimes \mathcal{B}.$$

3.  $Pf^{-1}(f(X)) = P(X) = 1$ .
4.  $f^{-1}(A \times (B \cap \varphi(X)) \times C) = A \cap \varphi^{-1}(B) \cap \varphi^{-1}(C) \in \mathcal{A}$  for all  $A \in \mathcal{A}, B \in \mathcal{B}$  and  $C \in \mathcal{B}$ .

5. For  $A \in \mathcal{A}, f(A) = (A \times \varphi(X) \times [0, 1]) \cap f(X) \in \mathcal{A} \otimes (\mathcal{B} \cap \varphi(X)) \otimes \mathcal{B}$ .

Hence, we have

$$\begin{aligned} (X, \mathcal{A}, P) &\xleftarrow{f} (f(X), (\mathcal{A} \otimes (\mathcal{B} \cap \varphi(X)) \otimes \mathcal{B}) \cap f(X), Pf^{-1}) \\ &\subset (X \times \varphi(X) \times [0, 1], \mathcal{A} \otimes (\mathcal{B} \cap \varphi(X)) \otimes \mathcal{B}, Pf^{-1}) \\ &\stackrel{def}{=} (X_1, \mathcal{A}_1, P_1). \end{aligned}$$

$P_1$  has marginals  $P, Q_0$  and  $\overline{Q}_0$  respectively on  $\mathcal{A}, \mathcal{B} \cap \varphi(X)$  and  $\mathcal{B}$  (by 4 above). Define

$$X_2 = [0, 1] \times (\varphi(X))^c, \quad \mathcal{A}_2 = \mathcal{B} \otimes (\mathcal{B} \cap (\varphi(X))^c).$$

Before defining  $P_2$  observe that  $\overline{Q}_0^*((\varphi(X))^c) = 1 - \alpha > 0$  and  $\overline{Q}_{0*}((\varphi(X))^c) = 0$ . Choose and fix  $B_0 \in \mathcal{B}, B_0 \subset \varphi(X)$  such that  $\overline{Q}_0(B_0) = \overline{Q}_{0*}(\varphi(X)) = \alpha$ . Let  $Q_1$  on  $\mathcal{B}$  be defined by

$$Q_1(B) = \overline{Q}_0(B|B_0^c) = \frac{\overline{Q}_0(B - B_0)}{1 - \alpha}, \quad B \in \mathcal{B}.$$

Then  $Q_1(B_0^c) = 1$  and  $B_0^c \supset (\varphi(X))^c$ . If  $B \in \mathcal{B}$  with  $B \supset (\varphi(X))^c$ , then  $B^c \subset \varphi(X) \Rightarrow \overline{Q}_0(B^c - B_0) = 0 \Rightarrow Q_1(B^c) = 0 \Rightarrow Q_1(B) = 1$  and so  $Q_1^*((\varphi(X))^c) = 1$ . Similarly, it can be seen that  $Q_{1*}((\varphi(X))^c) = 0$ . Now define  $P_2$  on  $\mathcal{A}_2$  by

$$P_2(A_2) = Q_1^*(\pi_2(A_2 \cap \Delta)), A_2 \in \mathcal{A}_2.$$

Marginals of  $P_2$  are  $Q_1$  and  $Q_1^*$  respectively. Having constructed suitable  $(X_i, \mathcal{A}_i, P_i), i = 1, 2$ , with  $(X, \mathcal{A}, P) \subset (X_1, \mathcal{A}_1, P_1)$ , we now show that  $(X_1, \mathcal{A}_1, P_1)$  fails to be a duality space.

Consider

$$X_1 \times X_2 = (X \times \varphi(X) \times [0, 1]) \times ([0, 1] \times (\varphi(X))^c)$$

and define

$$H = X \times \varphi(X) \times \Delta \times (\varphi(X))^c \in \mathcal{A}_1 \otimes \mathcal{A}_2.$$

Let  $h = 1_H$ . Since  $P_1(X \times (\Delta \cap (\varphi(X) \times [0, 1]))) = 1 = P_2(\Delta \cap X_2)$ , for every  $\mu \in \mathcal{M}(P_1, P_2)$  we get  $\mu(X \times (\Delta \cap (\varphi(X) \times [0, 1])) \times (\Delta \cap X_2)) = 1$ . But  $H \cap (X \times (\Delta \cap (\varphi(X) \times [0, 1])) \times (\Delta \cap X_2)) = \emptyset$  and so  $\mu(H) = 0$  for all  $\mu \in \mathcal{M}(P_1, P_2)$ . Hence  $S(h) = 0$ .

By (3.3) of Kellerer [4] (see also Strassen [12]), we have

$$I(h) = \inf\{P_1(A_1) + P_2(A_2) : H \subset (A_1 \times X_2) \cup (X_1 \times A_2), A_i \in \mathcal{A}_i, i = 1, 2\}.$$

Since  $H \subset (\emptyset \times X_2) \cup (X_1 \times X_2)$ , we get

$$I(h) \leq P_1(\emptyset) + P_2(X_2) = 1.$$

If  $H \subset (A_1 \times X_2) \cup (X_1 \times A_2)$  with  $A_i \in \mathcal{A}_i, i = 1, 2$ , then let

$$B_2 = \pi_2(A_2 \cap \Delta) \in \mathcal{B} \cap (\varphi(X))^c.$$

Hence,  $B_2 = B \cap (\varphi(X))^c$  for some  $B \in \mathcal{B}$  and so

$$(1) \quad P_2(A_2) = Q_1^*(B_2) = Q_1(B) = \frac{\overline{Q}_0(B - B_0)}{1 - \alpha}.$$

Carefully look at

$$D = X \times \varphi(X) \times (B^c \cap (\varphi(X))^c).$$

$$\begin{aligned} (x, y_1, y_2) \in D &\Rightarrow (x, y_1, y_2, y_2, y_2) \in H \\ &\Rightarrow (x, y_1, y_2, y_2, y_2) \in A_1 \times X_2 \\ &\quad ((y_2, y_2) \notin A_2 \text{ since } y_2 \notin B_2) \\ &\Rightarrow (x, y_1, y_2) \in A_1. \end{aligned}$$

It follows that  $D \subset A_1$ . Hence

$$(2) \quad P_1(A_1) \geq P_1^*(D) = \overline{Q}_0^*(B^c \cap (\varphi(X))^c) = \overline{Q}_0(B^c - B_0).$$

(To conclude that  $\overline{Q}_0^*(C \cap (\varphi(X))^c) = \overline{Q}_0(C - B_0)$  for all  $C \in \mathcal{B}$  first note that  $\overline{Q}_0(C - B_0) \geq \overline{Q}_0^*(C \cap (\varphi(X))^c)$  since  $B_0 \subset \varphi(X)$ ; on the other hand, if  $C \cap (\varphi(X))^c \subset C_1 \in \mathcal{B}$ , then  $(C - B_0) - (C_1 - B_0) \subset \varphi(X) - B_0 \Rightarrow \overline{Q}_0((C - B_0) - (C_1 - B_0)) = 0 \Rightarrow \overline{Q}_0(C - B_0) = \overline{Q}_0((C - B_0) \cap (C_1 - B_0)) \leq \overline{Q}_0(C_1 - B_0) \leq \overline{Q}_0(C_1)$ .)

Adding (1) and (2),

$$\begin{aligned} P_1(A_1) + P_2(A_2) &\geq \frac{\overline{Q}_0(B - B_0)}{1 - \alpha} + \overline{Q}_0(B^c - B_0) \\ &\geq \overline{Q}_0(B - B_0) + \overline{Q}_0(B^c - B_0) \\ &= \overline{Q}_0(B_0^c) = 1 - \alpha. \end{aligned}$$

Thus  $I(h) \geq 1 - \alpha > 0$  and as a consequence  $(X_1, \mathcal{A}_1, P_1)$  fails to be a duality space. □

Our main result which yields a new characterization of perfectness of measures is

**Theorem 3.** *Let  $(X, \mathcal{A}, P)$  be a probability space. Then the following statements are equivalent:*

- (a)  $(X, \mathcal{A}, P)$  is perfect.
- (b) If  $(X, \mathcal{A}, P) \subset (X_1, \mathcal{A}_1, P_1)$ , then  $(X_1, \mathcal{A}_1, P_1)$  is perfect.
- (c) If  $(X, \mathcal{A}, P) \subset (X_1, \mathcal{A}_1, P_1)$ , then  $(X_1, \mathcal{A}_1, P_1)$  is a duality space.

*Proof.* (a)  $\Rightarrow$  (b). If  $f_1$  is an  $\mathcal{A}_1$ -measurable, real-valued function on  $X_1$ , then letting  $f = f_1|_X$ , choose a Borel set  $B_f$  of the real line such that  $B_f \subset f(X) \subset f_1(X_1)$  with  $P(f^{-1}B_f) = 1$ . Take  $B_{f_1} = B_f$ .

(b)  $\Rightarrow$  (c) follows from Theorem 2.

(c)  $\Rightarrow$  (a) is Proposition 1. □

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