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HERMITE MULTIPLIERS AND PSEUDO-MULTIPLIERS

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ABSTRACT. We prove a multiplier theorem for the Hermite-Triebel-Lizorkin spaces introduced by Epperson in [Studia Math. **114** (1995), 87–103]. This extends Thangavelu's theorem [Revist. Mat. Ibero **3** (1987), 1–24; Math. Notes, vol. 42, 1993] on Hermite multipliers for L^p spaces. We also prove an L^p boundedness result for a class of Hermite pseudo-multipliers.

1. INTRODUCTION AND MAIN RESULTS

We begin with a review of some of the notation and results from [1]. Consult [6] for background information on Hermite expansions. Let $h_k(x)$ denote the k^{th} $L^2(\mathbf{R})$-normalized Hermite function, $k \in \mathbf{N}_0 = \{0, 1, 2, \dots\}$. Recall that the collection $\{h_k\}_{k=0}^{\infty}$ is a complete orthonormal basis for L^2, and that $h_k(x)$ is an eigenfunction of the Hermite operator $H = -\frac{d^2}{dx^2} + x^2$ with corresponding eigenvalue $2k+1$. If $m : \mathbf{R}_+ \to \mathbf{C}$ is a bounded function, then we let $m(H)$ denote the bounded linear operator on L^2 defined by $m(H)h_k = m(2k+1)h_k$.

Now suppose $\varphi : \mathbf{R} \to \mathbf{C}$ is C^∞ and satisfies

$$\begin{array}{ll} \text{(i)} & \operatorname{supp}\varphi \subset [\frac{1}{2}, 2], \\ \text{(ii)} & |\varphi(x)| \geq c > 0 \text{ if } x \in [\frac{3}{4}, \frac{7}{4}]. \end{array}$$

For each $\mu \in \mathbf{N}_0$ define the operator $Q_\mu = \varphi(2^{-\mu}H)$. Let L^2_f denote the space of finite linear combinations of Hermite functions. For $g \in L^2_f$ define the *Hermite-Triebel-Lizorkin norm*

$$\|g\|_{H_p^{\alpha q}} = \|(\sum_{\mu=0}^{\infty}(2^{\mu\alpha}|Q_\mu g|)^q)^{1/q}\|_{L^p(\mathbf{R})}.$$

See [7, 8] for a detailed description of the Triebel-Lizorkin spaces which occur in Fourier analysis. The parameters α, q, p are assumed to satisfy $\alpha \in \mathbf{R}$, $1 < p < \infty$, and $1 < q \leq \infty$, with the usual interpretation if $q = \infty$. The space $H_p^{\alpha q}$ is defined to be the completion of L^2_f with respect to the $\|\cdot\|_{H_p^{\alpha q}}$ norm.

One of the main results in [1] is that the space $H_p^{\alpha q}$ is essentially independent of the particular choice of φ chosen to satisfy conditions (i), (ii). To be precise, suppose $\varphi^{(1)}$, $\varphi^{(2)}$ are two different C^∞ functions satisfying (i), (ii), and let $H_p^{\alpha q}(1)$, $H_p^{\alpha q}(2)$ denote the corresponding spaces. Then Theorem 1.1 of [1] states that $H_p^{\alpha q}(1)$ and $H_p^{\alpha q}(2)$ are identical as sets and have equivalent norms. Theorem 1.2 of [1]

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states that the spaces H_p^{02} and L^p are isomorphic and have equivalent norms, as is expected.

A function $m : \mathbf{R}_+ \to \mathbf{C}$ will be called an $H_p^{\alpha q}$ *Hermite multiplier* if the operator $m(H) : L_f^2 \to L_f^2$ has a bounded linear extension to $H_p^{\alpha q}$.

Theorem 1. *Let $\alpha \in \mathbf{R}$, $1 < p, q < \infty$. Suppose $m : \mathbf{R}_+ \to \mathbf{C}$ is bounded and satisfies $|m'(\kappa)| \leq c\kappa^{-1}$. Then m is an $H_p^{\alpha q}$ Hermite multiplier.*

Note that this is directly analogous to Mihlin's theorem [2] for Fourier multipliers. Thangavelu [5, 6] first proved this theorem for L^p spaces (the $\alpha = 0$, $q = 2$ case) using special g-functions based on the Hermite semigroup. Section 2 of this paper contains a natural, alternative approach to the proof of Theorem 1. Of course the derivative condition on m in Theorem 1 can be replaced by a difference condition. Let $\Delta m(2k+1) := m(2(k+1)+1) - m(2k+1)$. In the proof we only need m to satisfy $|\Delta m(2k+1)| \leq c(1+k)^{-1}$ for $k \in \mathbf{N}_0$, which is certainly implied by the condition given on m'.

Next we consider pseudo-multipliers. Let $a : \mathbf{R} \times \mathbf{R}_+ \to \mathbf{C}$ be bounded, and for $g \in L_f^2$ define

$$Ag(x) = \sum_{k=0}^{\infty} a(x, 2k+1)\langle g, h_k \rangle h_k(x). \tag{1}$$

Theorem 2. *Suppose $a(x, \kappa)$ is measurable in the x variable for each fixed κ, and satisfies $|\partial_\kappa^\gamma a(x,\kappa)| \leq c(1+\kappa)^{-\gamma}$ for $0 \leq \gamma \leq 5$. If the operator A is bounded on L^2, then A also extends to a bounded operator on L^p for $1 < p < 2$.*

Using a method from [1] we establish uniform weak-(L^1, L^1) bounds on certain truncated versions of A, from which Theorem 2 follows by Marcinkiewicz interpolation. See Section 3.

2. Multipliers

We begin by describing the main steps toward proving Theorem 1. As in [1], let $\psi : \mathbf{R} \to \mathbf{C}$ satisfy the same conditions (i), (ii) as φ, and the condition

$$\sum_{\mu=0}^{\infty} \varphi(2^{-\mu}x)\psi(2^{-\mu}x) = 1 \qquad \text{for all } x \geq 1.$$

Let $\rho(x) = \varphi(2x)\psi(2x) + \varphi(x)\psi(x) + \varphi(2^{-1}x)\psi(2^{-1}x)$. For $\mu \in \mathbf{N}_0$ define $T_\mu = \rho(2^{-\mu}H)$. Note that $Q_\mu = T_\mu Q_\mu$. Now let m be as in Theorem 1, and for each $\mu \in \mathbf{N}_0$ let $W_\mu = m(H)T_\mu$. Let $L^2(l^2)_f$ denote the subspace of $L^2(l^2)$ consisting of sequences $\{g_\mu\}_{\mu=0}^\infty$ such that only finitely many g_μ are nonvanishing. Finally, define

$$W : L^2(l^2)_f \to L^2(l^2)_f$$

by $W(\{g_\mu\}_{\mu=0}^\infty) = \{W_\mu g_\mu\}_{\mu=0}^\infty$. It is easy to see that for $g \in L_f^2$,

$$\|m(H)g\|_{H_p^{\alpha q}} = \|W(\{2^{\mu\alpha}Q_\mu g\})\|_{L^p(l^q)}.$$

Therefore, to prove Theorem 1 it suffices to show that W is bounded on $L^2(l^2)_f \cap L^p(l^q)$ in the $L^p(l^q)$ norm topology.

Lemma 2.1. *Let $1 < q < \infty$. Then W has a bounded linear extension to $L^q(l^q)$.*

Lemma 2.2. *Let $1 < q < \infty$. Then W is weak-$(L^1(l^q), L^1(l^q))$ bounded.*

By Marcinkiewicz interpolation, these two lemmas suffice to show that W is bounded on $L^2(l^2)_f \cap L^p(l^q)$ in the $L^p(l^q)$ norm topology, for $1 < p \leq q < \infty$. The case $1 < q \leq p < \infty$ follows from the facts that: (1) $L^{p'}(l^{q'})$ is the dual of $L^p(l^q)$, and (2) Lemmas 2.1 and 2.2 continue to hold if m and ρ are replaced by their complex conjugates.

The proofs of Lemmas 2.1 and 2.2 depend on integral estimates for the kernels of the W_μ operators. First we need

Lemma 2.3. *There exist constants $c_1, c_2 > 0$ independent of $L \geq 1$ such that*

$$\sum_{k=0}^{L} h_k^2(x) \leq c_1 L^{1/2} e^{-c_2 L^{-1} x^2}.$$

Proof. We recall the argument used to prove Lemma 3.2.1 in [6]. If $0 < r < 1$, then by Mehler's formula

$$\sum_{k=0}^{L} h_k^2(x) \leq r^{-L} \sum_{k=0}^{\infty} r^k h_k^2(x) = \pi^{-1/2} r^{-L} (1-r^2)^{-1/2} e^{-\frac{1-r}{1+r}x^2}.$$

Substituting $r = e^{-1/L}$ we get

$$\sum_{k=0}^{L} h_k^2(x) \leq c_1 L^{1/2} e^{-c_2 L^{-1} x^2}.$$

Lemma 2.4. *There exist constants $0 < c_1, c_2 < \infty$ such that for every $t > 0$, $\mu \in \mathbf{N}_0$, and $y \in \mathbf{R}$,*

$$\int_{|x-y|\geq t} |W_\mu(x,y)|\, dx \leq c_1 (2^{\mu/2} t)^{-1/2} e^{-c_2 2^{-\mu} y^2}. \tag{2}$$

Proof. Inequality (2) follows from

$$\int_{-\infty}^{\infty} (x-y)^2 |W_\mu(x,y)|^2\, dx \leq c_1 2^{-\mu/2} e^{-c_2 2^{-\mu} y^2} \tag{3}$$

by an application of Schwarz's inequality. To prove (3) we use a simple case of Thangavelu's Lemma 3.2.3 ([6], p. 72):

$$(x-y) W_\mu(x,y) = \frac{1}{2}(B-A)\Delta W_\mu(x,y). \tag{4}$$

Here $A = -\frac{\partial}{\partial x} + x$, $B = -\frac{\partial}{\partial y} + y$, and

$$\Delta W_\mu(x,y) := \sum_{k=0}^{\infty} \Delta(m(2k+1)\rho(2^{-\mu}(2k+1))) h_k(x) h_k(y).$$

Identity (4) is easily derived from the recursion relation

$$2x h_k(x) = (2k+2)^{1/2} h_{k+1}(x) + (2k)^{1/2} h_{k-1}(x),$$

together with the fact that

$$(-\tfrac{d}{dx} + x) h_k(x) = (2k+2)^{1/2} h_{k+1}(x).$$

Substituting (4) in (3), we get

(5)

$$\begin{aligned}
&\int_{-\infty}^{\infty}(x-y)^2|W_\mu(x,y)|^2\,dx\\
&\quad\le c\int_{-\infty}^{\infty}|B\Delta W_\mu(x,y)|^2\,dx + c\int_{-\infty}^{\infty}|A\Delta W_\mu(x,y)|^2\,dx\\
&\quad\le c\sum_{k=0}^{\infty}|\Delta m(2k+1)\rho(2^{-\mu}(2k+1)) + m(2k+3)\Delta\rho(2^{-\mu}(2k+1))|^2\\
&\qquad\cdot(2k+2)(h_{k+1}^2(y)+h_k^2(y))\\
&\quad\le c\sum_{k=0}^{\infty}(|(1+k)^{-1}\rho(2^{-\mu}(2k+1))| + |m(2k+3)2^{-\mu}\rho'(2^{-\mu}\xi(k))|)^2\\
&\qquad\cdot(2k+2)(h_{k+1}^2(y)+h_k^2(y)),
\end{aligned}$$

where each $\xi(k)$ is between $2k+1$ and $2k+3$. Since ρ is compactly supported away from the origin, there exist integers $0<N_1<N_2$ independent of $\mu\in\mathbf{N}_0$ such that the terms in (5) vanish unless $2^\mu N_1\le k\le 2^\mu N_2$. Thus (5) is bounded by

$$c\sum_{k=2^\mu N_1}^{2^\mu N_2+1}2^{-2\mu}2^\mu h_k^2(y)\le c_1 2^{-\mu/2}e^{-c_2 2^{-\mu}y^2},$$

by an application of Lemma 2.3.

Lemma 2.5. *There exists a constant $c<\infty$ such that for every $t>0$, $\mu\in\mathbf{N}_0$, and $y,z\in\mathbf{R}$ with $|y-z|\le t$,*

(6)

$$\int_{|x-z|\ge 2t}|W_\mu(x,y)-(W_\mu(x,z)+(y-z)D_2W_\mu(x,z))|\,dx\le c(2^{\mu/2}t)^{3/2}.$$

Proof. Let J denote the interval with endpoints y,z. We can rewrite the left side of (6) as

$$\begin{aligned}
&\int_{|x-z|\ge 2t}|\int_J(y-u)D_2^2W_\mu(x,u)du|\,dx\\
&\qquad\le\int_J|y-u|\int_{|x-z|\ge 2t}|D_2^2W_\mu(x,u)|\,dxdu\\
&\qquad\le t^2\sup_{u\in J}\int_{|x-z|\ge 2t}|D_2^2W_\mu(x,u)|\,dx.
\end{aligned}$$

We estimate the last integral using

$$D_2^2W_\mu(x,u)=(D_2^2-u^2)W_\mu(x,u)+u^2W_\mu(x,u)=I+II.$$

Note that

$$I=-2^\mu\sum_{k=0}^{\infty}m(2k+1)\rho(2^{-\mu}(2k+1))(2^{-\mu}(2k+1))h_k(x)h_k(y).$$

Hence, by the method of proving Lemma 2.4 we get

$$t^2\sup_{u\in J}\int_{|x-z|\ge 2t}|\,I\,|\,dx\le t^2\sup_{u\in J}\int_{|x-u|\ge t}|\,I\,|\,dx\le ct^2 2^\mu(2^{\mu/2}t)^{-1/2}.$$

Again using Lemma 2.4, we have

$$t^2 \sup_{u\in J} \int_{|x-z|\geq 2t} |\, II \,|\, dx \leq c(2^{\mu/2}t)^2 \sup_{u\in J} 2^{-\mu}u^2(2^{\mu/2}t)^{-1/2}e^{-c_2 2^{-\mu}u^2} \leq c(2^{\mu/2}t)^{3/2}.$$

Proof of Lemma 2.1. It suffices to show that the operators W_μ, $\mu \in \mathbf{N}_0$, are uniformly bounded on L^q. This is trivial for $q = 2$, so it suffices (by interpolation and duality) to show that the W_μ operators are uniformly weak-(L^1, L^1) bounded. To do this we must show that there exists a constant $c < \infty$ independent of $f \in L^1$, $\lambda > 0$, and $\mu \in \mathbf{N}_0$ such that

$$|\{x : |W_\mu f(x)| > \lambda\}| \leq \frac{c}{\lambda}\|f\|_{L^1}.$$

This will be a routine application of Lemmas 2.4 and 2.5. So fix $f \in L^1$ and $\lambda > 0$, and apply the Calderón-Zygmund lemma to get a collection of disjoint dyadic open intervals $\{I_j\}$ such that

(a) $|f(x)| \leq \lambda$ for a.e. $x \in \mathbf{R} \setminus \cup_j I_j$,
(b) $\sum_j |I_j| \leq \frac{1}{\lambda}\|f\|_{L^1}$,
(c) $\lambda \leq \frac{1}{|I_j|}\int_{I_j} |f(x)|\, dx \leq 2\lambda$ for all j.

Let z_j denote the centerpoint of I_j, and for $x \in I_j$ let

$$g(x) = \frac{1}{|I_j|}\int_{I_j} f(y)\, dy + \frac{12(x - z_j)}{|I_j|^3}\int_{I_j} f(y)(y - z_j)\, dy.$$

Also, if $x \in I_j$, let $b(x) = f(x) - g(x)$. For $x \notin \bigcup_j I_j$, let $g(x) = f(x)$ and $b(x) = 0$. Thus $f = g + b$ everywhere. Note that if $x \in I_j$, then $|g(x)| \leq 8\lambda$. Also, for a.e. $x \notin \bigcup_j I_j$, $|g(x)| \leq \lambda$.

By the standard argument $\|g\|_{L^2}^2 \leq c\lambda\|f\|_{L^1}$. So, by Chebyshev's inequality

$$|\{x : |W_\mu g(x)| > \lambda/2\}| \leq \frac{4}{\lambda^2}\|W_\mu g\|_{L^2}^2 \leq \frac{c}{\lambda}\|f\|_{L^1}.$$

Next we have to prove the correct sort of estimate for $|\{x : |W_\mu b(x)| > \lambda/2\}|$. Define $I_j^* = (z_j - |I_j|, z_j + |I_j|)$. Since $|\bigcup_j I_j^*| \leq \frac{2}{\lambda}\|f\|_{L^1}$, it suffices to estimate

$$|\{x \in \mathbf{R} \setminus \bigcup_j I_j^* : |W_\mu b(x)| > \lambda/2\}|.$$

For each j let $b_j = b \cdot \chi_{I_j}$. Then $b = \sum_j b_j$ a.e., $\int b_j(x)(x - z_j)\, dx = 0$, and $\int b_j(x)dx = 0$. By Chebyshev's inequality

$$|\{x \in \mathbf{R} \setminus \bigcup_j I_j^* : |W_\mu b(x)| > \lambda/2\}| \leq \frac{2}{\lambda}\sum_j \int_{\mathbf{R}\setminus I_j^*} |W_\mu b_j(x)|\, dx. \tag{7}$$

For each j define the kernel

$$L_\mu^j(x, y) = \begin{cases} W_\mu(x, y) & \text{if } 2^{\mu/2}|I_j| \geq 1, \\ W_\mu(x, y) - (W_\mu(x, z_j) + (y - z_j)D_2W_\mu(x, z_j)) & \text{if } 2^{\mu/2}|I_j| < 1. \end{cases}$$

Because of the vanishing moment conditions imposed on b_j, we have

$$\begin{aligned} \int_{\mathbf{R}\setminus I_j^*} |W_\mu b_j(x)|\, dx &= \int_{\mathbf{R}\setminus I_j^*} |\int_{I_j} L_\mu^j(x, y)b_j(y)dy|\, dx \\ &\leq \int_{I_j} |b_j(y)| \int_{\mathbf{R}\setminus I_j^*} |L_\mu^j(x, y)|dxdy. \end{aligned}$$

Now according to Lemmas 2.4 and 2.5 we see that (7) is bounded by

$$\frac{c}{\lambda}\sum_j \min\{(2^{\mu/2}|I_j|)^{-1/2},(2^{\mu/2}|I_j|)^{3/2}\}\int_{I_j}|b_j(y)|\,dy \le \frac{c}{\lambda}\|f\|_{L^1}.$$

Proof of Lemma 2.2. We need to show that there exists a constant $c<\infty$ independent of $\{f_\mu\}\in L^1(l^q)$, $\lambda>0$ such that

$$|\{x:(\sum_{\mu=0}^{\infty}|W_\mu f_\mu(x)|^q)^{1/q}>\lambda\}|\le\frac{c}{\lambda}\|\{f_\mu\}\|_{L^1(l^q)}.$$

So fix $\{f_\mu\}\in L^1(l^q)$ and $\lambda>0$, let $h(x)=(\sum_{\mu=0}^{\infty}|f_\mu(x)|^q)^{1/q}$, and apply the Calderón-Zygmund lemma to get a collection of disjoint open intervals $\{I_j\}$ such that

(a) $|h(x)|\le\lambda$ for a.e. $x\in\mathbf{R}\setminus\bigcup_j I_j$,
(b) $\sum_j|I_j|\le\frac{1}{\lambda}\|h\|_{L^1}$
(c) $\lambda\le\frac{1}{|I_j|}\int_{I_j}|h(x)|dx\le 2\lambda$ for all j.

Again let z_j denote the centerpoint of I_j, and for $x\in I_j$ let

$$g_\mu(x)=\frac{1}{|I_j|}\int_{I_j}f_\mu(y)dy+\frac{12(x-z_j)}{|I_j|^3}\int_{I_j}f_\mu(y)(y-z_j)dy.$$

For $x\in I_j$ let $b_\mu(x)=f_\mu(x)-g_\mu(x)$, and for $x\notin\bigcup_j I_j$ let $g_\mu(x)=f_\mu(x)$, $b_\mu(x)=0$. If $x\in I_j$ we have by Minkowski's inequality

$$\begin{aligned}
&(\sum_{\mu=0}^{\infty}|g_\mu(x)|^q)^{1/q}\\
&\quad\le(\sum_{\mu=0}^{\infty}|\frac{1}{|I_j|}\int_{I_j}f_\mu(y)dy|^q)^{1/q}+(\sum_{\mu=0}^{\infty}|\frac{12(x-z_j)}{|I_j|^3}\int_{I_j}f_\mu(y)(y-z_j)dy|^q)^{1/q}\\
&\quad\le\frac{1}{|I_j|}\int_{I_j}(\sum_{\mu=0}^{\infty}|f_\mu(y)|^q)^{1/q}dy+\frac{3}{|I_j|}\int_{I_j}(\sum_{\mu=0}^{\infty}|f_\mu(y)|^q)^{1/q}dy\\
&\quad\le 8\lambda.
\end{aligned}$$

It follows that $\|\{g_\mu\}\|^q_{L^q(l^q)}\le c\lambda^{q-1}\|\{f_\mu\}\|_{L^1(l^q)}$, and therefore by Chebyshev's inequality and Lemma 2.1,

$$|\{x:(\sum_{\mu=0}^{\infty}|W_\mu g_\mu(x)|^q)^{1/q}>\lambda/2\}|\le\frac{2^q}{\lambda^q}\|\{W_\mu g_\mu\}\|^q_{L^q(l^q)}\le\frac{c}{\lambda}\|\{f_\mu\}\|_{L^1(l^q)}.$$

Next we have to estimate $|\{x:(\sum_{\mu=0}^{\infty}|W_\mu b_\mu(x)|^q)^{1/q}>\lambda/2\}|$. As in the proof of Lemma 2.1 it suffices to handle

$$\begin{aligned}
&|\{x\in\mathbf{R}\setminus\bigcup_j I_j^*:(\sum_{\mu=0}^{\infty}|W_\mu b_\mu(x)|^q)^{1/q}>\lambda/2\}|\\
&\qquad\le\frac{2}{\lambda}\sum_j\int_{\mathbf{R}\setminus I_j^*}(\sum_{\mu=0}^{\infty}|W_\mu b_{\mu,j}(x)|^q)^{1/q}\,dx.
\end{aligned}\tag{8}$$

Here of course $b_{\mu,j} = b_\mu \cdot \chi_{I_j}$. Let L^j_μ be as in the proof of Lemma 2.1. Then by Minkowski's inequality and lemmas Lemmas 2.4 and 2.5

$$\begin{aligned}
&\int_{\mathbf{R}\setminus I_j^*} (\sum_{\mu=0}^{\infty} |W_\mu b_{\mu,j}(x)|^q)^{1/q}\, dx \\
&\quad = \int_{\mathbf{R}\setminus I_j^*} (\sum_{\mu=0}^{\infty} |\int_{I_j} L^j_\mu(x,y) b_{\mu,j}(y) dy|^q)^{1/q}\, dx \\
&\quad \le \int_{\mathbf{R}\setminus I_j^*} \int_{I_j} (\sum_{\mu=0}^{\infty} |L^j_\mu(x,y) b_{\mu,j}(y)|^q)^{1/q}\, dy dx \\
&\quad \le \int_{I_j} (\sum_{\mu=0}^{\infty} |b_{\mu,j}(y)|^q)^{1/q} (\int_{\mathbf{R}\setminus I_j^*} \sum_{\mu=0}^{\infty} |L^j_\mu(x,y)|\, dx)\, dy \\
&\quad \le c \int_{I_j} (\sum_{\mu=0}^{\infty} |b_{\mu,j}(y)|^q)^{1/q} (\sum_{\mu=0}^{\infty} \min\{(2^{\mu/2}|I_j|)^{-1/2}, (2^{\mu/2}|I_j|)^{3/2}\})\, dy \\
&\quad \le c \int_{I_j} (\sum_{\mu=0}^{\infty} |b_{\mu,j}(y)|^q)^{1/q}\, dy \\
&\quad \le c \int_{I_j} (\sum_{\mu=0}^{\infty} |f_\mu(y)|^q)^{1/q}\, dy
\end{aligned}$$

with c independent of $|I_j|$. Substituting this in (8) finishes the proof.

3. Pseudo-multipliers

In this section we prove Theorem 2. Let $P_\mu = \pi(2^{-\mu}H)$, where $\pi(x)$ is a C^∞ function supported on $[\frac{1}{2}, 2]$ with the property that $\sum_{\mu=0}^{\infty} \pi(2^{-\mu}x) = 1$ for all $x \ge 1$.

Lemma 3.1. *Let $a(x,\kappa)$ be as in the statement of Theorem 2, and suppose that the associated operator A is bounded on L^2. Then the operators $\sum_{\mu=0}^{N} A\,P_\mu$ are weak-(L^1, L^1) bounded, uniformly in N.*

Proof. We begin by recalling how to estimate the kernel $K_\mu(x,y)$ of the operator $A\,P_\mu$. Let $a_\mu(x,\kappa) = a(x,\kappa)\pi(2^{-\mu}\kappa)$, and let $\hat{a}_\mu(x,\xi)$ denote the Fourier transform of a_μ in its second variable. As in the derivation of (7) in §3 of [1], we have the representation

$$K_\mu(x,y) = c\int_{-\infty}^{\infty} \hat{a}_\mu(x,\xi/2) e^{i\xi/2}(1-e^{i2\xi})^{-1/2} e^{-\frac{i}{2}((x^2+y^2)\cot\xi - 2xy\csc\xi)} d\xi,$$

where c is some unimportant constant. Now the conditions on a imply that for every $l \in \mathbf{N}_0$ there exists a constant c_l independent of $\mu \in \mathbf{N}_0$ such that $|\partial^l_\xi \hat{a}_\mu(x,\xi)| \le c_l 2^{\mu(1+l)}(1+2^\mu|\xi|)^{-5}$. It follows by inspection of the proof of Lemma 1.1 in [1] that there exists a constant c independent of $\mu \in \mathbf{N}_0$ such that

$$|K_\mu(x,y)| \le \sum_{\sigma=\pm1} \frac{c2^{\mu/2}}{(1+2^{\mu/2}|x+\sigma y|)^4} \tag{9}$$

and

$$|\partial_y^2 K_\mu(x,y)| \le \sum_{\sigma=\pm1} \frac{c2^{3\mu/2}}{(1+2^{\mu/2}|x+\sigma y|)^2}. \tag{10}$$

The proof is finished with a simple modification of the proof of Lemma 2.1, which we very briefly indicate. Fix $f \in L^1$ and $\lambda > 0$, and let $\{I_j\}$, $g(x)$, $b(x)$, etc., be as in the proof of Lemma 2.1. From the L^2-boundedness of A and the uniform L^2-boundedness of the operators $\sum_{\mu=0}^N P_\mu$ we get

$$|\{x : |\sum_{\mu=0}^N A\,P_\mu g(x)| > \lambda/2\}| \leq \frac{c}{\lambda}\|f\|_{L^1}.$$

Now let $L_\mu^j(x,y)$ be defined as in the proof of Lemma 2.1, except with $K_\mu(x,y)$ in place of $W_\mu(x,y)$. Then from (9) and (10) we get

$$\begin{aligned}\int_{\mathbf{R}\setminus I_j^*} |\sum_{\mu=0}^N A\,P_\mu b_j(x)|\,dx &\leq \int_{I_j} |b_j(y)|(\int_{\mathbf{R}\setminus I_j^*} \sum_{\mu=0}^N |L_\mu^j(x,y)|\,dx)\,dy \\ &\leq c\int_{I_j} |b_j(y)|\,(\sum_{\mu=0}^\infty \min\{(2^{\mu/2}|I_j|)^{-3}, 2^{\mu/2}|I_j|\})\,dy \\ &\leq c\int_{I_j} |b_j(y)|\,dy.\end{aligned}$$

Proof of Theorem 2. By Lemma 3.1 and the Marcinkiewicz interpolation theorem, each of the operators $\sum_{\mu=0}^N A\,P_\mu$ has a bounded linear extension to L^p, for $1 < p \leq 2$. Moreover, the operator norms $\|\sum_{\mu=0}^N A\,P_\mu\|_{L^p \to L^p}$ are bounded uniformly in N. Now let $g \in L_f^2$. Then $Ag = \sum_{\mu=0}^N A\,P_\mu g$ for a large enough choice of N. Hence $\|Ag\|_{L^p} \leq c\|g\|_{L^p}$. The proof is finished by recalling that L_f^2 is dense in L^p (see for example [3], Lemma 2). It would be interesting to find natural criteria for the L^2-boundedness of a Hermite pseudo-multiplier, since the standard methods for obtaining L^2-boundedness of an ordinary pseudo-differential operator (as in [4]) do not seem to be applicable here.

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