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NON-COMMUTATIVE DISC ALGEBRAS AND THEIR REPRESENTATIONS

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Abstract. It is shown that the smallest closed subalgebra

$$Alg(I_{\mathcal{K}}, V_1, \dots, V_n) \subset \mathcal{B}(\mathcal{K}) \qquad (n = 2, 3, \dots, \infty)$$

generated by any sequence $V_1, \dots, V_n$ of isometries on a Hilbert space $\mathcal{K}$ such that $V_1V_1^* + \cdots + V_nV_n^* \le I_{\mathcal{K}}$ is completely isometrically isomorphic to the non-commutative "disc" algebra $\mathcal{A}_n$ introduced in Math. Scand. **68** (1991), 292–304. We also prove that for $n \neq m$ the Banach algebras $\mathcal{A}_n$ and $\mathcal{A}_m$ are not isomorphic. In particular, we give an example of two non-isomorphic Banach algebras which are completely isometrically embedded in each other.

The completely bounded (contractive) representations of the "disc" algebras $\mathcal{A}_n$ $(n = 2, 3, \dots, \infty)$ on a Hilbert space are characterized. In particular, we prove that a sequence of operators $A_1, A_2, \dots$ is simultaneously similar to a contractive sequence $T_1, T_2, \dots$ (i.e., $T_1T_1^* + \cdots + T_nT_n^* \le I$) if and only if it is completely polynomially bounded.

The first cohomology group of $\mathcal{A}_n$ with coefficients in $\mathbb{C}$ is calculated, showing, in particular, that the disc algebras are not amenable. Similar results are proved for the non-commutative Hardy algebras F_n^∞ introduced in Math. Scand. **68** (1991), 292–304.

The right joint spectrum of the left creation operators on the full Fock space is also determined.

1. Introduction and preliminaries

Let $\mathcal{H}$ be a Hilbert space and $B(\mathcal{H})$ the set of bounded linear operators on $\mathcal{H}$. We identify $M_m(B(\mathcal{H}))$, the set of $m \times m$ matrices with entries from $B(\mathcal{H})$, with $B(\underbrace{\mathcal{H} \oplus \cdots \oplus \mathcal{H}}_{\text{m-times}})$. Thus we have a natural C^*-norm on $M_m(B(\mathcal{H}))$. If X is an operator space, i.e., a closed subspace of $B(\mathcal{H})$, we consider $M_m(X)$ as a subspace of $M_m(B(\mathcal{H}))$ with the induced norm. Let X, Y be operator spaces and $u : X \to Y$ a linear map. Define $u_m : M_m(X) \to M_m(Y)$ by

$$u_m\left[(x_{ij})\right] = \left[(u(x_{ij}))\right].$$

We say that u is completely bounded (cb in short) if

$$\|u\|_{cb} = \sup_{m \ge 1} \|u_m\| < \infty.$$

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If $\|u\|_{cb} \leq 1$ (resp. u_m is an isometry for any $m \geq 1$), then u is completely contractive (resp. isometric), and if u_m is positive for all m, then u is called completely positive. In the following we fix $n = 1, 2, 3, \ldots, \infty$.

Let us consider the full Fock space [E]

$$F^2(H_n) = \mathbf{C}1 \oplus \bigoplus_{m \geq 1} H_n^{\otimes m}$$

where H_n is an n-dimensional complex Hilbert space with orthonormal basis $\{e_1, e_2, \ldots, e_n\}$ if n is finite, or $\{e_1, e_2, \ldots\}$ if $n = \infty$. For each $i = 1, 2, \ldots$, $S_i \in B(F^2(H_n))$ is the left creation operator with e_i, i.e., $S_i \xi = e_i \otimes \xi$, $\xi \in F^2(H_n)$. We shall denote by $\mathcal{P}_n$ the set of all $p \in F^2(H_n)$ of the form

$$p = a_0 + \sum_{\substack{1 \leq i_1, \ldots, i_k \leq n \\ 1 \leq k \leq m}} a_{i_1 \cdots i_k} e_{i_1} \otimes \cdots \otimes e_{i_k}, \quad m \in \mathbb{N}, \tag{1.1}$$

where $a_0, a_{i_1 \ldots i_k} \in \mathbf{C}$ and the sum contains only a finite number of summands. The set $\mathcal{P}_n$ may be viewed as the algebra of polynomials in n non-commuting indeterminates, with $p \otimes q$, $p, q \in \mathcal{P}_n$, as multiplication. Define F_n^∞ as the set of all $g \in F^2(H_n)$ such that

$$\|g\|_\infty := \sup\{\|g \otimes p\|_2 : p \in \mathcal{P}_n, \|p\|_2 \leq 1\} < \infty$$

where $\|\cdot\| = \|\cdot\|_{F^2(H_n)}$. $(F_n^\infty, \|\cdot\|_\infty)$ is a non-commutative Banach algebra [Po3]. We denote by $\mathcal{A}_n$ the closure of $\mathcal{P}_n$ in $(F_n^\infty, \|\cdot\|_\infty)$. The Banach algebra F_n^∞ (resp. $\mathcal{A}_n$) can be viewed as a non-commutative analogue of the Hardy space H^∞ (resp. disc algebra); when $n = 1$ they coincide.

Let $(B(\mathcal{H})^n)_1$ denote the unit ball of $(B(\mathcal{H})^n)_1$, i.e.,

$$(B(\mathcal{H})^n)_1 = \{(T_1, \ldots, T_n) \in B(\mathcal{H})^n : \sum_{i=1}^n T_i T_i^* \leq I_{\mathcal{H}}\}.$$

For any sequence $T_1, T_2, \ldots, T_n \in B(\mathcal{H})$ and $p \in \mathcal{P}_n$ given by (1.1) we denote by $p(T_1, \ldots, T_n)$ the operator acting on $\mathcal{H}$, defined by

$$p(T_1, \ldots, T_n) = a_0 I_{\mathcal{H}} + \sum a_{i_1 \ldots i_k} T_{i_1} \cdots T_{i_k}.$$

The von Neumann inequality [vN],[SzF] for $(B(\mathcal{H})^n)_1$ (see [Po3]) asserts that if $(T_1, \ldots, T_n) \in (B(\mathcal{H})^n)_1$ and $p \in \mathcal{P}_n$, then

$$\|p(T_1, \ldots, T_n)\| \leq \|p(S_1, \ldots, S_n)\| = \|p\|_\infty. \tag{1.2}$$

According to [Po3] the mapping

$$\Psi : \mathcal{A}_n \to B(\mathcal{H}); \quad \Psi(f) = f(T_1, \ldots, T_n)$$

is a contractive homomorphism.

A sequence $A_1, \ldots, A_n \in B(\mathcal{H})$ will be called completely polynomially bounded (in short c.p.b.) if there is a constant C such that for all m and all $m \times m$ matrices (p_{ij}) with $p_{ij} \in \mathcal{P}_n$ we have

$$\|(p_{ij}(A_1, \ldots, A_n))\|_{M_m(B(\mathcal{H}))} \leq C \|(p_{ij})\|_{M_m(\mathcal{A}_n)}.$$

Here of course we consider $\mathcal{A}_n$ as a subalgebra of the C^*-algebra $C^*(S_1, \ldots, S_n)$ (see [Po3]). Notice that $A_1, \ldots, A_n$ is c.p.b. if and only if the homomorphism $p \to p(A_1, \ldots, A_n)$ defines a completely bounded homomorphism from the "disc" algebra $\mathcal{A}_n$ into $B(\mathcal{H})$. All the above considerations hold true for $n = \infty$ in a slightly adapted version (see also [Po3]).

2. Completely bounded representations of $\mathcal{A}_n$

It is well known that an operator $T \in B(\mathcal{H})$ is a contraction if and only if it gives rise to a completely contractive representation of the classical disc algebra. In what follows we get an extension of this result to our non-commutative setting.

Theorem 2.1. *Let $A_1, \ldots, A_n$ be in $B(\mathcal{H})$. Then $[A_1, \ldots, A_n]$ is a contraction if and only if the map*

$$\Phi : \mathcal{P}_n \to B(\mathcal{H}); \quad \Phi(p) = p(A_1, \ldots, A_n)$$

is completely contractive.

Proof. Assume that $[A_1, \ldots, A_n]$ is a contraction. According to [Po2] there is a sequence $\{V_1, \ldots, V_n\}$ of isometries on a Hilbert space $\mathcal{K} \supset \mathcal{H}$ such that

$$\sum_{i=1}^{n} V_i V_i^* \leq I_{\mathcal{K}} \text{ and } V_i^*|_{\mathcal{H}} = A_i^*, \; i = 1, 2, \ldots, n. \tag{2.1}$$

Using the result from [Po5], the map $\Psi : \mathcal{P}_n \to C^*(V_1, \ldots, V_n)$ defined by

$$\Psi(p) = p(V_1, \ldots, V_n) \tag{2.2}$$

can be extended to a $*$-representation of $C^*(S_1, \ldots, S_n)$ by setting

$$\Phi(S_{i_1} \cdots S_{i_k} S_{j_1}^* \cdots S_{j_p}^*) = V_{i_1} \cdots V_{i_k} V_{j_1}^* \cdots V_{j_p}^*,$$

where $1 \leq i_1, \ldots, i_k, j_1, \ldots, j_p \leq n$, and by extending it by linearity. In particular we have $\|\Psi\|_{cb} \leq 1$. According to (2.1) we have

$$\Phi(p) = p(A_1, \ldots, A_n) = P_{\mathcal{H}} p(V_1, \ldots, V_n)|_{\mathcal{H}},$$

which together with (2.2) implies

$$\Phi(f) = f(A_1, \ldots, A_n) = P_{\mathcal{H}} f(V_1, \ldots, V_n)|_{\mathcal{H}} = P_{\mathcal{H}} \Psi(f)|_{\mathcal{H}},$$

for all $f \in \mathcal{A}_n$. Therefore $\|\Phi\|_{cb} \leq \|\Psi\|_{cb} \leq 1$.

Conversely, suppose that $A_1, \ldots, A_n \in B(\mathcal{H})$ such that the map

$$\Phi : \mathcal{P}_n \to B(\mathcal{H}); \quad \Phi(p) = p(A_1, \ldots, A_n)$$

is completely contractive. In particular we have

$$\left\| \begin{bmatrix} A_1 & A_2 & \cdots & A_n \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & 0 \end{bmatrix} \right\| \leq \left\| \begin{bmatrix} S_1 & S_2 & \cdots & S_n \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & 0 \end{bmatrix} \right\|.$$

Hence, $\| \sum_{i=1}^{n} A_i A_i^* \| \leq \| \sum_{i=1}^{n} S_i S_i^* \| = 1$. This completes the proof. ∎

Let us remark that if $A_1, A_2, \dots, A_n \in B(\mathcal{H})$ such that the map

$$\Phi : \mathcal{P}_n \to B(\mathcal{H}); \quad \Phi(p) = p(A_1, \dots, A_n)$$

is completely contractive, one can get another proof for the existence of an isometric dilation for $A_1, A_2, \dots, A_n$ (see [Po2]).

Indeed, according to [Arv, Prop. 1.2.8] Φ has a unique completely positive extention $\tilde{\Phi}$ to the closure of $\mathcal{P}_n + \mathcal{P}_n^*$ such that

$$\tilde{\Phi}(p + q^*) = p(A_1, \dots, A_n) + q(A_1, \dots, A_n)^* \tag{2.3}$$

for any $p, q \in \mathcal{P}_n \subset C^*(S_1, \dots, S_n)$. Here $\mathcal{P}_n^*$ stands for $\{p(S_1, \dots, S_n)^*; p \in \mathcal{P}_n\}$. Using the extension theorem of Arveson [Arv] we infer that there is a completely positive linear map $\Psi : C^*(S_1, \dots, S_n) \to B(\mathcal{H})$ such that

$$\Psi|_{\overline{\mathcal{P}_n + \mathcal{P}_n^*}} = \tilde{\Phi}. \tag{2.4}$$

According to Stinespring's theorem [S]

$$\Psi(f) = V^* \pi(f) V, \quad \text{for any} \quad f \in C^*(S_1, \dots, S_n), \tag{2.5}$$

where π is a $*$-representation of $C^*(S_1, \dots, S_n)$ on some Hilbert space $\mathcal{K}$ and V is a bounded operator from $\mathcal{H}$ to $\mathcal{K}$. Since $\Psi(1) = I$, it follows that $V^*V = I$, that is, V is an isometric embedding of $\mathcal{H}$ in $\mathcal{K}$. Using V, we can identify $\mathcal{H}$ with a subspace of $\mathcal{K}$, and (2.5) becomes $\Psi(f) = P_{\mathcal{H}} \pi(f)|_{\mathcal{H}}$, $P_{\mathcal{H}}$ being the projection of $\mathcal{K}$ on $\mathcal{H}$. Since

$$\pi(S_i)^* \pi(S_j) = \pi(S_i^* S_j) = \begin{cases} 0, & \text{if } i \neq j, \\ I_{\mathcal{K}}, & \text{if } i = j, \end{cases}$$

it follows that $\{\pi(S_i)\}_{i=1}^n$ are isometries with orthogonal ranges. On the other hand, the relations (2.3), (2.4), (2.5) imply

$$p(A_1, \dots, A_n) = P_{\mathcal{H}} p(\pi(S_1), \dots, \pi(S_n))|_{\mathcal{H}}, \quad \text{for any } i = 1, 2, \dots, n \text{ and } p \in \mathcal{P};$$

that is, $\{\pi(S_i)\}_{i=1}^n$ is an isometric dilation of $\{A_i\}_{i=1}^n$.

Corollary 2.2. $[A_1, \dots, A_n]$ *is a contraction if and only if the map*

$$\Phi : C^*(S_1, \dots, S_n) \to B(\mathcal{H}); \quad \Phi(S_{i_1} \cdots S_{i_k} S_{j_1}^* \cdots S_{j_p}^*) = A_{i_1} \cdots A_{i_k} A_{j_1}^* \cdots A_{j_p}^*,$$

$1 \leq i_1, \dots, i_k, j_1, \dots, j_p \leq n$, *is a completely contractive linear map.*

Corollary 2.3. $[A_1, \dots, A_n]$ *is a contraction if and only if there is a completely positive linear map* $\Phi : C^*(S_1, \dots, S_n) \to B(\mathcal{H})$ *such that* $\Phi(S_i) = A_i$, *for* $i = 1, 2, \dots, n$.

In [P1] V. Paulsen proved that an operator $A \in B(\mathcal{H})$ is similar to a contraction if and only if it is completely polynomially bounded. In what follows we will extend this result to our setting.

Theorem 2.4. *Let $A_1, \dots, A_n$ be in $B(\mathcal{H})$. The following statements are equivalent:*

(i) *There exist a contraction $[T_1, \dots, T_n]$ and an invertible operator S such that*

$$A_i = S^{-1} T_i S, \quad \textit{for any } i = 1, 2, \dots, n.$$

(ii) *The sequence $A_1, \dots, A_n$ is c.p.b.*

Proof. Assume (i) holds. We have

$$\Phi(p) = p(A_1, \dots, A_n) = Sp(T_1, \dots, T_n)S^{-1} \text{ for any } \ p \in \mathcal{P}_n.$$

Since the map

$$\Psi : \mathcal{P}_n \to B(\mathcal{H}); \quad \Psi(p) = p(T_1, \dots, T_n)$$

is completely contractive by Theorem 2.1, it follows that

$$\|\Phi\|_{cb} \leq \|S\| \|\Psi\|_{cb} \|S^{-1}\| < \infty.$$

Conversely, assume that (ii) holds. Since $\Phi : \mathcal{A}_n \to B(\mathcal{H}); \quad \Phi(f) = f(A_1, \dots, A_n)$ is completely bounded, according to [P1] there is an invertible operator S such that

$$\mathcal{A}_n \in f \to S\Phi(f)S^{-1} = f(SA_1S^{-1}, \dots, SA_nS^{-1}) \in B(\mathcal{H})$$

is completely contractive. Now, Theorem 2.1 shows that $\left[SA_1S^{-1}, \dots, SA_nS^{-1}\right]$ is a contraction. Setting $SA_iS^{-1} = T_i, \ i = 1, 2, \dots, n$, the proof is complete. □

Corollary 2.5. *A representation $\Phi : \mathcal{A}_n \to B(\mathcal{H})$ is c.b. if and only if it is given by $\Phi(S_i) = ST_iS^{-1}, i = 1, \dots, n$, where $[T_1, \dots, T_n]$ is a contraction and S is an invertible operator.*

Let $A_1, \dots, A_n$ be a sequence of operators on $\mathcal{H}$. Following [B2] we define the "spectral radius" of this sequence as being

$$r(A_1, \dots, A_n) := \lim_{k \to \infty} \| \sum_{|f|=k} A_f A_f^* \|^{1/2k},$$

where for any $f = (i_1, \dots, i_k)$, $1 \leq i_1, \dots, i_k \leq n$, A_f stands for the product $A_{i_1} \cdots A_{i_k}$ and $|f| = k$.

Proposition 2.6. *Let $A_1, \dots, A_n$ be in $B(\mathcal{H})$. If the map*

$$\Phi : \mathcal{P}_n \to B(\mathcal{H}); \ \Phi = p(A_1, \dots, A_n)$$

is c.b., then $r(A_1, \dots, A_n) \leq 1$.

Proof. According to [Po1, Prop.3.5]

$$r(A_1, \dots, A_n) = \inf_S \| \sum_{i=1}^n (SA_iS^{-1})(SA_iS^{-1})^* \|^{1/2}$$

where the infimum is taken over all invertible operators on $\mathcal{H}$. By Theorem 2.4 there exist a contraction $[T_1, \dots, T_n]$ and an invertible operator S such that $A_i = S^{-1}T_iS$ for any $i = 1, 2, \dots, n$. Therefore, $r(A_1, \dots, A_n) \leq \| \sum T_iT_i^* \|^{1/2} \leq 1$. □

Propositon 2.7. *Let $A_1, \dots, A_n \in B(\mathcal{H})$ such that one of the following statements holds:*

(i) *There exists $b > a > 0$ such that*

$$a\|h\|^2 \le \sum_{|f|=k} \|A_f^* h\|^2 \le b\|h\|^2 \quad \text{for any } h \in \mathcal{H} \text{ and } k = 1, 2, \dots .$$

(ii) $r(A_1, \dots, A_n) < 1$.

Then, the map $\Phi : \mathcal{P}_n \to B(\mathcal{H})$; $\Phi(p) = p(A_1, \dots, A_n)$ is c.b.

Proof. If either one of the above statements holds, then, according to [Po1], there exist a contraction $[T_1, \dots, T_n]$ and an invertible operator S such that $A_i = S^{-1}T_iS$, $i = 1, 2, \dots, n$. According to Theorem 2.4 the result follows. □

Let us remark that if (i) holds, then $r(A_1, \dots, A_n) = 1$. All the results of this section can be easily extended to $n = \infty$.

3. The non-commutative disc algebras $\mathcal{A}_n$

Let $V_1, \dots, V_n$ be isometries on a Hilbert space $\mathcal{K}$ such that $V_1V_1^* + \dots + V_nV_n^* \le I_{\mathcal{K}}$. Let $Alg(V_1, \dots, V_n)$ denote the smallest closed subalgebra of $B(\mathcal{K})$ containing $I_{\mathcal{K}}, V_1, \dots, V_n$. This algebra is the closure in the uniform norm of the collection of polynomials in $V_1, \dots, V_n$, that is,

$$Alg(V_1, \dots, V_n) = clos\{p(V_1, \dots, V_n) : \ p \in \mathcal{P}_n\}.$$

In [Po3] we proved that if $S_1, \dots, S_n$ are the left creation operators on the Fock space $F^2(H_n)$, then the Banach algebras $Alg(S_1, \dots, S_n)$ and $\mathcal{A}_n$ are isometrically isomorphic. In what follows we will obtain a more general result.

Theorem 3.1. *The Banach algebras $Alg(V_1, \dots, V_n)$ and $\mathcal{A}_n$ ($n = 2, 3, \dots, \infty$) are completely isometrically isomorphic.*

Proof. Consider the case $n = 2, 3, \dots$. Suppose first that $V_1, \dots, V_n$ are isometries on $\mathcal{K}$ such that $V_1V_1^* + \dots + V_nV_n^* = I_{\mathcal{K}}$. According to the von Neumann inequality (1.2) we have

$$\|p(V_1, \dots, V_n)\| \le \|p(S_1, \dots, S_n)\|, \text{ for any } p \in \mathcal{P}_n. \tag{3.1}$$

On the other hand, according to [Po2, Prop.2.6] there is a sequence $W_1, \dots, W_n$ of isometries on a Hilbert space $\mathcal{G} \supset \mathcal{K}$ such that

$$\sum_{i=1}^n W_iW_i^* = I_{\mathcal{G}} \quad \text{and} \quad S_{i_1} \cdots S_{i_k} = P_{\mathcal{K}} W_{i_1} \cdots W_{i_k}|_{\mathcal{K}}$$

for any $1 \le i_1, \dots, i_k \le n$. Hence we deduce that

$$p(S_1, \dots, S_n) = P_{\mathcal{K}} p(W_1, \dots, W_n)|_{\mathcal{K}}, \quad p \in \mathcal{P}_n,$$

and

$$\|p(S_1, \dots, S_n)\| \le \|p(W_1, \dots, W_n)\|. \tag{3.2}$$

Since the Cuntz algebra $\mathcal{O}_n$ does not depend on the generators [Cu], we have that

$$\|p(V_1, \ldots, V_n)\| = \|p(W_1, \ldots, W_n)\|, \tag{3.3}$$

which together with (3.1) and (3.2) shows that

$$\|p(V_1, \ldots, V_n)\| = \|p(S_1, \ldots, S_n)\|, \text{ for any } p \in \mathcal{P}_n. \tag{3.4}$$

In the case when $V_1V_1^* + \cdots + V_nV_n^* \leq I_{\mathcal{K}}$, the relation (3.4) holds also according to [Po5]. This relation shows that the map $V_i \to S_i$ $(i = 1, 2, \ldots, n)$ extends to an isometric isomorphism from $Alg(V_1, \ldots, V_n)$ onto $\mathcal{A}_n$. The completely isometric (in short c.i.) part follows in the same way, passing to matrices. The case $n = \infty$ follows from [Cu]. This completes the proof. $\square$

Let $V_1, V_2, \ldots$ be a sequence of isometries satisfying $\sum_{i=1}^{k} V_iV_i^* \leq I$ for every $k \in \mathbb{N}$. According to Theorem 3.1 we have $Alg(V_1, \ldots, V_k) \overset{\text{c.i.}}{\simeq} \mathcal{A}_k$ for any $k = 1, 2, \ldots$ and $Alg(V_1, V_2, \ldots) \overset{\text{c.i.}}{\simeq} \mathcal{A}_\infty$. Thus, it is clear that

$$\mathcal{A}_2 \overset{\text{c.i.}}{\subset} \mathcal{A}_3 \overset{\text{c.i.}}{\subset} \cdots \overset{\text{c.i.}}{\subset} A_\infty. \tag{3.5}$$

On the other hand, consider $\mathcal{A}_2 = Alg(S_1, S_2)$. If we put

$$V_1 = S_1, \; V_2 = S_2S_1, \ldots, V_k = S_2^{k-1}S_1, \ldots,$$

then $\sum_{i=1}^{k} V_iV_i^* \leq I$ for every $k \in \mathbb{N}$. By Theorem 3.1 we have

$$\mathcal{A}_\infty \overset{\text{c.i.}}{\simeq} Alg(V_1, V_2, \ldots) \subset Alg(S_1, S_2) \overset{\text{c.i.}}{\simeq} \mathcal{A}_2 \text{ and } \mathcal{A}_3 \overset{\text{c.i.}}{\simeq} Alg(V_1, V_2, V_3) \subset \mathcal{A}_2.$$

By induction we get the following chain of inclusions:

$$\mathcal{A}_2 \overset{\text{c.i.}}{\supset} A_3 \overset{\text{c.i.}}{\supset} \cdots \overset{\text{c.i.}}{\supset} \mathcal{A}_\infty. \tag{3.6}$$

In [A] A. Arias showed that, as Banach spaces, the disc algebras are completely isomorphic. In what follows we will show that they are not isomorphic as Banach algebras.

Let $\lambda = (\lambda_1, \ldots, \lambda_n)$ be in $\overline{(\mathbb{C}^n)}_1 = \{(\lambda_1, \ldots . \lambda_n) : |\lambda_1|^2 + \cdots + |\lambda_n|^2 \leq 1\}$, and define the "evaluation" functional

$$\Phi_\lambda : \mathcal{P}_n \to \mathbb{C}; \quad \Phi_\lambda(p) = p(\lambda_1, \ldots, \lambda_n). \tag{3.7}$$

According to the von Neumann inequality (1.2) we have

$$|p(\lambda_1, \ldots, \lambda_n)| = \|p(\lambda_1 I_{\mathbb{C}}, \ldots, \lambda_n I_{\mathbb{C}})\| \leq \|p(S_1, \ldots, S_n)\| = \|p\|_\infty.$$

Hence, Φ_λ has a unique extension to the disc algebra $\mathcal{A}_n$. Therefore Φ is a character of $\mathcal{A}_n$. Let $M_{\mathcal{A}_n}$ be the set of all characters of $\mathcal{A}_n$ and let $\Psi : \overline{(\mathbb{C}^n)}_1 \to M_{\mathcal{A}_n}$ be defined by $\Psi(\lambda) = \Phi_\lambda$.

Theorem 3.2. *Ψ is a homeomorphism of $\overline{(\mathbb{C}^n)}_1$ onto $M_{\mathcal{A}_n}$* $(n = 2, 3, \dots)$.

Proof. Let us show that Ψ is one-to-one. If $\lambda = (\lambda_1, \dots, \lambda_n)$ and $\mu = (\mu_1, \dots, \mu_n)$ are in $\overline{(\mathbb{C}^n)}_1$, then $\Psi(\lambda) = \Psi(\mu)$ implies that

$$\lambda_i = \Phi_\lambda(S_i) = \Phi_\mu(S_i) = \mu_i, \text{ for any } i = 1, 2, \dots, n.$$

Therefore $\lambda = \mu$. Now, assume that $\Phi : \mathcal{A}_n \to \mathbb{C}$ is a character. Setting $\Phi(S_i) = \lambda_i,\ i = 1, 2, \dots, n$, we have

$$\Phi(p(S_1, \dots, S_n)) = p(\lambda_1, \dots, \lambda_n), \text{ for any } p \in \mathcal{P}_n.$$

Since Φ is a character, it follows that it is completely contractive. Applying Theorem 2.1 when $A_i = \lambda_i I_{\mathbb{C}},\ i = 1, 2, \dots, n$, we infer that $[\lambda_1 I_{\mathbb{C}}, \dots, \lambda_n I_{\mathbb{C}}]$ is a contraction, i.e., $|\lambda_1|^2 + \cdots + |\lambda_n|^2 \leq 1$. Moreover the identity

$$\Phi(p(S_1, \dots, S_n)) = p(\lambda_1, \dots, \lambda_n) = \Phi_\lambda(p(S_1, \dots, S_n))$$

proves that Φ agrees with Φ_λ on the dense subset $\mathcal{P}_n$ of $\mathcal{A}_n$, therefore $\Phi = \Phi_\lambda$.

Since both $\overline{(\mathbb{C}^n)}_1$ and $M_{\mathcal{A}_n}$ are compact Hausdorff spaces and Ψ is one-to-one and onto, to complete the proof it suffices to show that Ψ is continuous.

Suppose that $\lambda^\alpha = (\lambda_1^\alpha, \dots, \lambda_n^\alpha),\ \alpha \in J$, is a net in $\overline{(\mathbb{C}^n)}_1$ such that $\lim_{\alpha \in J} \lambda^\alpha = \lambda = (\lambda_1, \dots, \lambda_n)$. Since $\sup_{\alpha \in J} \|\Phi_{\lambda^\alpha}\| \leq 1$ and $\mathcal{P}_n$ is dense in $\mathcal{A}_n$, and since

$$\lim_{\alpha \in J} \Phi_{\lambda^\alpha}(p(S_1, \dots, S_n)) = \lim_{\alpha \in J} p(\lambda_1^\alpha, \dots, \lambda_n^\alpha) = p(\lambda_1, \dots, \lambda_n) = \Phi_\lambda(p(S_1, \dots, S_n))$$

for every $p \in \mathcal{P}_n$, it follows that Ψ is continuous. The proof is complete. □

Remark 3.3. The above theorem can be easily extended to the case $n = \infty$ in a slightly adapted version, showing that $\overline{(\ell^2)}_1$ (with the weak topology) is homeomorphic to $M_{\mathcal{A}_\infty}$.

The Cuntz algebra $\mathcal{O}_n$ is uniquely defined as the C^*-algebra generated by $n = 2, 3, \dots$ isometries satisfying

$$s_i^* s_j = \delta_{ij} 1, \quad \sum_{j=1}^{n} s_i s_j^* = 1 \tag{3.8}$$

[Cu]. We define $\mathcal{O}_1 = C(\mathbb{T})$ (see [C]), and $\mathcal{O}_\infty$ as the C^*-algebra generated by isometries $s_1, s_2, \dots$ satisfying merely the first relation in (3.8). Notice that the disc algebra $\mathcal{A}_n$ can be viewed as a subalgebra of the Cuntz algebra $\mathcal{O}_n$. Indeed, if $s_1, \dots, s_n$ is a system of generators for $\mathcal{O}_n$, then according to Theorem 3.1 $\mathcal{A}_n \overset{\text{c.i.}}{\simeq} Alg(s_1, \dots, s_n) \subset \mathcal{O}_n$ $(n = 2, 3, \dots)$ and $\mathcal{A}_\infty \overset{\text{c.i.}}{\simeq} Alg(s_1, s_2, \dots) \subset \mathcal{O}_\infty$.

It was proved by Pimsner and Popa [PP] that if $n \neq m$, then $\mathcal{O}_n \not\simeq \mathcal{O}_m$ (see also [PS]). Using Theorem 3.2 and the dimension theory [HW] it is easy to get the following.

Corollary 3.4. *The Banach algebras $\mathcal{A}_n$ and $\mathcal{A}_m$ are not isomorphic if $n \neq m$, $n, m = 1, 2, \dots, \infty$.*

On the other hand, since $\mathcal{O}_n$ has no non-trivial character, it follows that $\mathcal{A}_n \not\simeq \mathcal{O}_m$ for any $n, m = 2, \dots, \infty$.

4. Disc algebras and cohomology

Let A be a complex Banach algebra with unit, X a Banach A-bimodule, and X' the dual Banach A-bimodule (see [BD]). We need to recall from [BD] a few definitions.

A bounded X-derivation is a bounded linear mapping D of A into X such that

$$D(ab) = (Da)b + a(Db), \quad \text{for any } a, b \in A. \tag{4.1}$$

The set of all bounded X-derivations is denoted by $Z^1(A, X)$. For each $x \in X$ let us define $\delta_x : A \to X$ by $\delta_x(a) = ax - xa$. We call δ_x an inner X-derivation, and denote by $B^1(A, X)$ the set of all inner X-derivations. The quotient space $Z^1(A, X)/B^1(A, X)$ is called the first cohomology group of A with coefficients in X, and it is denoted by $H^1(A, X)$. A Banach algebra A is said to be amenable if $H^1(A, X') = \{0\}$ for every Banach A-bimodule X.

In what follows we shall show that the disc algebras $\mathcal{A}_n$ $(n = 2, 3, \dots, \infty)$ are not amenable.

Of course $\mathbb{C}$, the set of all complex numbers, is a Banach $\mathcal{A}_n$-bimodule under the module multiplication

$$\lambda \cdot f = f \cdot \lambda = \lambda f(0), \tag{4.2}$$

where for each $f \in \mathcal{A}_n$, $f(0) := \Phi_{(0,\dots,0)}(f)$ (see the relation (3.7)). Notice that $|\lambda \cdot f| \le |\lambda| \|f\|_\infty$, for any $\lambda \in \mathbb{C}$, $f \in \mathcal{A}_n$.

Theorem 4.1. *The first cohomology group of $\mathcal{A}_n$ $(n = 2, 3, \dots)$ with coefficients in $\mathbb{C}$ is isomorphic to the additive group $\mathbb{C}^n$, i.e., $H^1(\mathcal{A}_n, \mathbb{C}) \simeq (\mathbb{C}^n, +)$. If $n = \infty$, then $H^1(\mathcal{A}_\infty, \mathbb{C}) \simeq (\ell^2, +)$.*

Proof. It is clear that $B^1(\mathcal{A}_n, \mathbb{C}) = \{0\}$. If $D \in Z^1(\mathcal{A}_n, \mathbb{C})$, then, using (4.1) and (4.2) it is easy to see that $D(1) = 0$ and $D(e_{i_1} \otimes \cdots \otimes e_{i_k}) = 0$ for $k = 2, 3, \dots$. Therefore the derivation D is determined by $D(e_1), D(e_2), \dots, D(e_n)$. Let $D(e_i) = \lambda_i \in \mathbb{C}$, $i = 1, 2, \dots, n$. Since D is a linear mapping for each $f \in \mathcal{A}_n, f = a_0 + \sum_{i=1}^n a_i e_i + \cdots$, we have $D(f) = \sum_{i=1}^n a_i \lambda_i$. It is easy to see that

$$D(f \otimes g) = D(f)g + fD(g), \quad \text{for any } f, g \in \mathcal{A}_n.$$

Let us show that D is bounded if and only if $\sum_{i=1}^n |\lambda_i|^2 < \infty$ $(n = 2, 3, \dots, \infty)$. For each $f \in \mathcal{A}_n, f = a_0 + \sum_{i=1}^n a_i e_i + \cdots$, we have

$$\begin{aligned} |D(f)| = |\sum_{i=1}^n a_i \lambda_i| &\le \left(\sum_{i=1}^n |a_i|^2 \right)^{1/2} \left(\sum_{i=1}^n |\lambda_i|^2 \right)^{1/2} \\ &\le \|f\|_2 \left(\sum_{i=1}^n |\lambda_i|^2 \right)^{1/2} \le \|f\|_\infty \left(\sum_{i=1}^n |\lambda_i|^2 \right)^{1/2}. \end{aligned}$$

Therefore D is a bounded $\mathbb{C}$-derivation. We need to prove the converse only for $n=\infty$. Suppose that $D:\mathcal{A}_\infty\to\mathbb{C}$ is a bounded derivation. For each $f\in\mathcal{A}_\infty, f=a_0+\sum_{i=1}^n a_ie_i+\cdots,\ D(f)=\sum_{i=1}^\infty a_i\lambda_i$ for some $\lambda_i\in\mathbb{C}, i=1,2,\dots,$ and

$$|D(f)|\le M\|f\|_\infty,\quad \text{for any } f\in\mathcal{A}_\infty. \tag{4.3}$$

In particular, for any $\{a_i\}_{i=1}^\infty\in\ell^2, g=\sum_{i=1}^\infty a_ie_i$ is in $\mathcal{A}_\infty$ and $\|g\|_\infty=\|g\|_2$. In this case the relation (4.3) shows that

$$|\sum_{i=1}^\infty a_i\lambda_i|\le M\|\{a_i\}_{i=1}^\infty\|_2,$$

for any $\{a_i\}_{i=1}^\infty\in\ell^2$. Hence we deduce that $\{\lambda_i\}_{i=1}^\infty\in\ell^2$.

Now it is clear that $H^1(\mathcal{A}_n,\mathbb{C})\simeq(\mathbb{C}^n,+)$ for $n=2,3,\dots,$ and $H^1(\mathcal{A}_\infty,\mathbb{C})\simeq(\ell^2,+)$. □

Since $\mathbb{C}$ is a dual bimodule, we have the following.

Corollary 4.2. *The disc algebras $\mathcal{A}_n$ $(n=2,3,\dots,\infty)$ are not amenable.*

A similar proof to that of Theorem 4.1 shows the following.

Remark 4.3. $H^1(F_n^\infty,\mathbb{C})\simeq(\mathbb{C}^n,+)$ for $n=2,3,\dots,$ and $H^1(F_\infty^\infty,\mathbb{C})\simeq(\ell^2,+)$.

Corollary 4.4. *The non-commutative Hardy algebras F_n^∞ $(n=2,3,\dots,\infty)$ are not amenable. Moreover, if $n\ne m, n,m=1,2,\dots,\infty$, then F_n^∞ and F_m^∞ are not Banach isomorphic.*

5. The right joint spectrum of $(S_1,\dots,S_n)$

If $A=(A_1,\dots,A_n)$ is an n-tuple of operators acting on $\mathcal{H}$, then the joint left (resp. right) spectrum of A is the set of n-tuples of complex numbers $\lambda=(\lambda_1,\lambda_2,\dots,\lambda_n)$ such that the left (resp. right) ideal of $B(\mathcal{H})$ generated by the set $\{\lambda_1I-A_1,\lambda_2I-A_2,\dots,\lambda_nI-A_n\}$ does not contain the identity operator (see [B1], [B2]). Let us denote the left (resp. right) spectrum of A by $\sigma_l(A)$ (resp. $\sigma_r(A)$). Let $S_1,\dots,S_n$ be the left creation operators on the full Fock space $F^2(H_n)$.

Theorem 5.1. $\sigma_r(S_1,\dots,S_n)=\overline{(\mathbb{C}^n)}_1$.

Proof. Let $\mu=(\mu_1,\dots,\mu_n)\in\overline{(\mathbb{C}^n)}_1$. Suppose that there is $\delta>0$ such that

$$\sum_{i=1}^n\|(S_i-\mu_iI)^*h\|^2\ge\delta\|h\|^2,\quad \text{for any } h\in F^2(H_n). \tag{5.1}$$

For

$$h=1+\sum_{k=1}^\infty\sum_{1\le i_1,\dots,i_k\le n}\lambda_{i_1}\cdots\lambda_{i_k}e_{i_1}\otimes\cdots\otimes e_{i_k},$$

where $(\lambda_1,\dots,\lambda_2)\in(\mathbb{C}^n)_1$, we have $S_i^*h=\lambda_ih,\ \ i=1,2,\dots,n$, and

$$\sum_{i=1}^n\|(S_i-\mu_iI)^*h\|^2=\sum_{i=1}^n|\lambda_i-\overline{\mu}_i|^2\|h\|^2.$$

The relation (5.1) becomes

$$\sum_{i=1}^{n} |\lambda_i - \overline{\mu}_i|^2 \geq \delta, \quad \text{for any } (\lambda_1, \ldots, \lambda_n) \in (\mathbb{C}^n)_1. \tag{5.2}$$

Since (5.2) is not true, according to the Corona-type theorem [Po4] it does not exist $\{B_1, \ldots, B_n\} \subset B(F^2(H_n))$ such that

$$(S_1 - \mu_1 I)B_1 + \cdots + (S_n - \mu_n I)B_n = I.$$

Therefore $\overline{(\mathbb{C}^n)}_1 \subset \sigma_r(S_1, \ldots, S_n)$.

Conversely, let $(\lambda_1, \ldots, \lambda_n) \in \sigma_r(S_1, \ldots, S_n)$. This means that the left ideal of $B(F^2(H_n))$ generated by $\{S_1^* - \overline{\lambda}_1 I, \ldots, S_n^* - \overline{\lambda}_n I\}$ does not contain the identity. According to [D, Theorem 2.9.5] there is a state σ on $B(F^2(H_n))$ such that $\sigma(XS_i^*) = \overline{\lambda}_i \sigma(X)$ for each $X \in B(F^2(H_n))$ and $i = 1, 2, \ldots, n$. Then, for any $k = 1, 2, \ldots$,

$$|\sigma(\sum_{|f|=k} S_f S_f^*)| \leq \| \sum_{|f|=k} S_f S_f^* \| = 1.$$

Hence, $\sum_{|f|=k} \|\lambda_f\|^2 \leq 1$. Since $(\sum_{j=1}^{n} |\lambda_j|^2)^k = \sum_{|f|=k} |\lambda_f|^2$, it follows that $(\lambda_1, \ldots, \lambda_n) \in \overline{(\mathbb{C}^n)}_1$. This completes the proof. □

Using Theorem 3.2 we infer the following.

Remark 5.2. (1) $\sigma_r(S_1, \ldots, S_n) = \{(\Phi(S_1), \ldots, \Phi(S_n)) : \Phi \in M_{\mathcal{A}_n}\}$.

(2) If $f_1, \ldots, f_n \in \mathcal{A}_n$, then $\sigma_r(f_1, \ldots, f_n) \supset \{(\Phi(f_1), \ldots, \Phi(f_n)) : \Phi \in M_{\mathcal{A}_n}\}$.

6. Open problems

Let $\lambda = (\lambda_1, \ldots, \lambda_n) \in (\mathbb{C}^n)_1$ and let $\Phi_\lambda : F_n^\infty \to \mathbb{C}$ be defined by $\Phi(f) = \langle f, z_\lambda \rangle$, $f \in F^\infty$, where $z_\lambda = 1 + \sum_{k=1}^{\infty} (\lambda_1 e_1 + \cdots + \lambda_n e_n)^k$. One can prove that Φ_λ is a multiplicative functional on F_n^∞.

Problem 6.1. *Is $\{\Phi_\lambda : \lambda \in (\mathbb{C}^n)_1\}$ w^*-dense in the set of all multiplicative functionals on F_n^∞ ?*

In [Arv] W. Arveson proved that if T is a contraction on a Hilbert space, then T gives rise to a completely isometric representation of the disc algebra if and only if the spectrum of T contains the unit disc.

Problem 6.2. *Characterize the contractions $[T_1, \ldots, T_n]$ for which the map*

$$\mathcal{A}_n \in p \mapsto p(T_1, \ldots, T_n) \in B(\mathcal{H})$$

is a completely isometric representation of the disc algebra $\mathcal{A}_n$.

According to [Po3], [Po5] and Theorem 3.1, one can show that if $T_i = A_i \oplus V_i$, where V_i, $i = 1, 2, \ldots, n$, are isometries, then the above map is a completely isometric representation.

Let $\Phi : \mathcal{A}_n \to B(\mathcal{H})$ defined by $\Phi(p) = p(A_1, \ldots, A_n)$, where $A_1, \ldots, A_n \in B(\mathcal{H})$.

Problem 6.3. *Is the implication, Φ contractive $\implies$ Φ completely contractive, true*?

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