

## MULTINOMIAL EXPANSIONS AND THE PYTHAGOREAN THEOREM

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(Communicated by J. Marshall Ash)

*Dedicated to the memory of Professor Morris Marden*

ABSTRACT. An addition formula for homogeneous polynomials is used to obtain a generalization of the Pythagorean theorem and a new view of the multinomial expansion.

Let  $x = (x_1, x_2, \dots, x_n) \in \mathbf{R}^n$ , and let  $P$  denote the set of all polynomials

$$p(x) = \sum_{j=1}^m a_j x_1^{\alpha_j^1} x_2^{\alpha_j^2} \dots x_n^{\alpha_j^n}, \quad a_j \in \mathbf{R}.$$

Letting  $\partial/\partial x = (\partial/\partial x_1, \partial/\partial x_2, \dots, \partial/\partial x_n)$ , we define on  $P$  the inner product

$$\begin{aligned} (p, q) &= p \left( \frac{\partial}{\partial x} \right) q(x) \Big|_{x=0} \\ (1) \quad &= \left( \sum_{j=1}^m a_j \frac{\partial^{\alpha_j^1}}{\partial x_1^{\alpha_j^1}} \frac{\partial^{\alpha_j^2}}{\partial x_2^{\alpha_j^2}} \dots \frac{\partial^{\alpha_j^n}}{\partial x_n^{\alpha_j^n}} \right) q(x) \Big|_{x=0}. \end{aligned}$$

Let  $H_k \subset P$  denote the vector space of homogeneous polynomials of degree  $k$ . Note that the dimension of  $H_k$  is

$$\dim H_k = d_k = \frac{(n+k-1)!}{(n-1)! k!}.$$

We first obtain a simple analog of the Funk-Hecke theorem [1, p. 247] for homogeneous polynomials.

**Theorem 1.** *If  $p(x)$  is a homogeneous polynomial of degree  $k$ , then*

$$((x_1 y_1 + x_2 y_2 + \dots + x_n y_n)^k, p(y)) = k! p(x).$$

*Proof.* Expanding the power,

$$\begin{aligned} (x_1 y_1 + \dots + x_n y_n)^k &= \sum_{\alpha} \frac{k!}{\alpha_1! \alpha_2! \dots \alpha_n!} (x_1 y_1)^{\alpha_1} (x_2 y_2)^{\alpha_2} \dots (x_n y_n)^{\alpha_n} \\ &= k! \sum_{\alpha} \frac{x_1^{\alpha_1} x_2^{\alpha_2} \dots x_n^{\alpha_n}}{\alpha_1! \alpha_2! \dots \alpha_n!} y_1^{\alpha_1} y_2^{\alpha_2} \dots y_n^{\alpha_n}. \end{aligned}$$

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Received by the editors October 7, 1994.

1991 *Mathematics Subject Classification.* Primary 33E99; Secondary 26C99.

*Key words and phrases.* Homogeneous polynomials, addition formula.

Thus if  $p(y) = \sum_{\alpha} a_{\alpha} y_1^{\alpha_1} y_2^{\alpha_2} \dots y_n^{\alpha_n}$ , we have

$$\begin{aligned} & ((x_1 y_1 + \dots + x_n y_n)^k, p(y)) \\ &= \left( k! \sum_{\alpha} \frac{x_1^{\alpha_1} \dots x_n^{\alpha_n}}{\alpha_1! \dots \alpha_n!} \frac{\partial^{\alpha_1}}{\partial y_1^{\alpha_1}} \dots \frac{\partial^{\alpha_n}}{\partial y_n^{\alpha_n}} \sum_{\beta} a_{\beta} y_1^{\beta_1} \dots y_n^{\beta_n} \right) \\ &= k! \sum_{\alpha} a_{\alpha} x_1^{\alpha_1} \dots x_n^{\alpha_n} = k! p(x). \end{aligned}$$

Our next result is an addition formula for orthonormal homogeneous polynomials.

**Theorem 2** (Addition Formula). *Let  $\{p_j(x)\}_{j=1}^{d_k}$  be an orthonormal basis for  $H_k$ . Then*

$$(2) \quad (x_1 y_1 + x_2 y_2 + \dots + x_n y_n)^k = k! \sum_{j=1}^{d_k} p_j(x) p_j(y).$$

*Proof.* Let  $x = (x_1, x_2, \dots, x_n)$  and  $\alpha_j = (\alpha_j^1, \alpha_j^2, \dots, \alpha_j^n)$ . For convenience we write  $x^{\alpha_j} = x_1^{\alpha_j^1} x_2^{\alpha_j^2} \dots x_n^{\alpha_j^n}$ . Then expanding the power, we have

$$(x_1 y_1 + \dots + x_n y_n)^h = \sum c_j x^{\alpha_j} y^{\alpha_j},$$

where  $x^{\alpha_j}$  and  $y^{\alpha_j}$  are homogeneous monomials of degree  $k$ . Since  $\{p_j(x)\}_{j=1}^{d_k}$  is a basis for  $H_k$ , there exist constants  $a_{jm}$  such that

$$y^{\alpha_j} = \sum_{m=1}^{d_k} a_{jm} p_m(y).$$

Thus

$$\begin{aligned} (x_1 y_1 + \dots + x_n y_n)^k &= \sum_j c_j x^{\alpha_j} \left( \sum_m a_{jm} p_m(y) \right) \\ &= \sum_j g_j(x) p_j(y), \end{aligned}$$

where the last expression is a rearrangement of the previous sum. The  $g_j(x)$  are linear combinations of the monomials  $x^{\alpha_j}$ , and thus are homogeneous polynomials of degree  $k$  in  $x$ . Further, since the  $p_j(x)$  are orthonormal,

$$((x_1 y_1 + \dots + x_n y_n)^k, p_l(y)) = \left( \sum_j g_j(x) p_j(y), p_l(y) \right) = g_l(x).$$

But by the identity given in Theorem 1,

$$((x_1 y_1 + \dots + x_n y_n)^k, p_l(y)) = k! p_l(x).$$

Thus,

$$g_j(x) = k! p_j(x), \quad j = 1, 2, \dots, d_k,$$

which completes the proof.

If  $\alpha_j = (\alpha_j^1, \alpha_j^2, \dots, \alpha_j^n)$ , for convenience we write  $\alpha_j! = \alpha_j^1! \alpha_j^2! \dots \alpha_j^n!$ . The result of Theorem 2 shows that the multinomial expansion

$$(x_1y_1 + x_2y_2 + \dots + x_ny_n)^k = \sum \frac{k!}{\alpha_j!} x^{\alpha_j} y^{\alpha_j} = k! \sum \frac{x^{\alpha_j}}{\sqrt{\alpha_j!}} \frac{y^{\alpha_j}}{\sqrt{\alpha_j!}}$$

is no more than an addition formula for the orthonormal basis  $\{x^{\alpha_j}/\sqrt{\alpha_j!}\}_{j=1}^{d_k}$  of  $H_k$ . The significance of the result (2) lies with the fact that it holds for any orthonormal basis  $\{p_j(x)\}_1^{d_k}$ , not merely the monomials  $\{x^{\alpha_j}/\sqrt{\alpha_j!}\}_{j=1}^{d_k}$ .

The addition formula of Theorem 2 yields an identity which contains the Pythagorean theorem  $\sin^2 \theta + \cos^2 \theta = 1$  as a special case:

**Theorem 3** (Pythagorean Identity). *Suppose  $\{p_j(x)\}_{j=1}^{d_k}$  is an orthonormal basis for  $H_k$ . Then on the unit sphere  $|s| = \sqrt{s_1^2 + s_2^2 + \dots + s_n^2} = 1$ ,*

$$\sum_{j=1}^{d_k} [p_j(s)]^2 = \frac{1}{k!}.$$

*Proof.* Letting  $x = y$  in the addition formula yields

$$(x_1^2 + x_2^2 + \dots + x_n^2)^k = k! \sum_{j=1}^{d_k} [p_j(x)]^2,$$

from which the result is immediate.

Most inner products are integrals. Although the inner product we have used here is defined by a derivative, it is far from artificial. Indeed, this inner product allows us to view Taylor series as Fourier series. That is, suppose we have the Taylor series

$$f(x) = \sum_{j=0}^{\infty} a_j x^{\alpha_j}.$$

Rewriting the series as

$$f(x) = \sum_{j=1}^{\infty} b_j \frac{x^{\alpha_j}}{\sqrt{\alpha_j!}},$$

and noting that the monomials  $\{x^{\alpha_j}/\sqrt{\alpha_j!}\}_{j=1}^{\infty}$  are orthonormal with respect to the inner product (1), we have

$$(x^{\alpha_j}/\sqrt{\alpha_j!}, f(x)) = b_j.$$

These simple results appear to have considerable analytic significance. To see this, one need only consider the kernel

$$\sum_{k=0}^{\infty} \sum_{j=1}^{d_k} p_k^j(x) p_k^j(y) = \sum_{k=0}^{\infty} \frac{(x \cdot y)^k}{k!} = e^{x \cdot y},$$

and the reproducing formula

$$(f(y), e^{x \cdot y}) = f(x).$$

Here  $\{p_k^j(x)\}_{j=1}^{d_k}$ ,  $k = 0, 1, 2, \dots$ , are arbitrary orthonormal bases for the homogeneous polynomials  $H_k$  of degree  $k$ .

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