

A TOPOLOGICAL CHARACTERIZATION OF LINEARITY FOR QUASI-TRACES

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ABSTRACT. Let \mathcal{A} be a C^* -algebra, and let μ be a (local) quasi-trace on \mathcal{A} . Then μ is linear if, and only if, the restriction of μ to the closed unit ball of \mathcal{A} is uniformly weakly continuous.

INTRODUCTION

Throughout this paper, \mathcal{A} will be a C^* -algebra and A will be the real Banach space of selfadjoint elements of \mathcal{A} . The closed unit ball of \mathcal{A} is \mathcal{A}_1 and the closed unit ball of A is A_1 . We do not assume the existence of a unit in \mathcal{A} .

Definition. A quasi-linear functional on \mathcal{A} is a function $\mu : \mathcal{A} \rightarrow \mathbb{C}$ with the following properties:

- (1) μ is bounded on \mathcal{A}_1 .
- (2) Whenever a and b are in A then

$$\mu(a + ib) = \mu(a) + i\mu(b).$$

- (3) Whenever \mathcal{B} is an abelian $*$ -subalgebra of \mathcal{A} then the restriction of μ to \mathcal{B} is linear.

Examples of non-linear, quasi-linear functionals on $M_2(\mathbb{C})$ exist in great abundance. See [7] for a full description of all the non-linear quasi-states on $M_2(\mathbb{C})$.

Definition. A quasi-linear functional, μ , on \mathcal{A} is said to be a *quasi-trace* if, whenever x and y are in \mathcal{A} ,

$$\mu(xy) = \mu(yx).$$

We do *not* require a quasi-trace to be positive on positive elements of A .

Handelman [12], Blackadar and Handelman [4] and Haagerup [11] have made powerful contributions to the problem of determining when quasi-traces are linear. They consider bi-traces, that is, positive quasi-traces which are postulated to be well-behaved with respect to tensoring with $M_2(\mathbb{C})$, the algebra of two-by-two complex matrices.

In this note, we impose no algebraic restrictions on the quasi-traces considered. Instead, we show that for a quasi-trace, linearity is equivalent to uniform weak continuity on the unit ball.

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Our proofs are brief because most of the hard work has been done elsewhere. In particular, we proved in [6] that:

Theorem. *Let \mathcal{A} be a C^* -algebra with no quotient isomorphic to $M_2(\mathbb{C})$. Let μ be a quasi-linear functional on \mathcal{A} . Then μ is linear if, and only if, μ is uniformly weakly continuous on \mathcal{A}_1 .*

This result is best possible, in the sense that, if \mathcal{A} does have a quotient onto $M_2(\mathbb{C})$, then there exists a quasi-linear functional μ on \mathcal{A} , where μ is uniformly weakly continuous on \mathcal{A} , and μ is not linear; see [8]. However, when μ is a quasi-trace, we shall show that the analogue of the above theorem is true for *all* \mathcal{A} .

In fact, we shall use a weaker property than quasi-linearity, local quasi-linearity.

Definition. A function $\mu : \mathcal{A} \rightarrow \mathbb{C}$ is said to be *locally quasi-linear* if the following properties hold:

- (1) μ is bounded on \mathcal{A}_1 .
- (2) Whenever a and b are in \mathcal{A} then

$$\mu(a + ib) = \mu(a) + i\mu(b).$$

- (3) Whenever \mathcal{B} is a closed $*$ -subalgebra of \mathcal{A} generated by a single Hermitian element then the restriction of μ to \mathcal{B} is linear.

It is clear that when μ is quasi-linear, then μ is locally quasi-linear. Surprisingly, the converse is false, even when \mathcal{A} is abelian and separable. Aarnes [2] constructs a counter-example where \mathcal{A} is the C^* -algebra of continuous functions on the unit square. However, whenever \mathcal{A} is sufficiently rich in projections, in particular when \mathcal{A} is a von Neumann algebra, then local quasi-linearity does imply quasi-linearity [3].

Definition. A function $\mu : \mathcal{A} \rightarrow \mathbb{C}$ is a *local quasi-trace* if μ is locally quasi-linear and, for each x and y in \mathcal{A} ,

$$\mu(xy) = \mu(yx).$$

1. TYPE I_n ALGEBRAS

Let S be a compact Hausdorff space, and let $C(S)$ be the commutative C^* -algebra of all continuous complex-valued functions on S . The selfadjoint part of $C(S)$ is a boundedly complete vector lattice if, and only if, S is extremally disconnected, that is, the closure of each open subset of S is also open. In all that follows we shall assume S to be extremally disconnected.

Let $M_n(C(S))$ be the algebra of $n \times n$ matrices over $C(S)$. Then

$$M_n(C(S)) = C(S, M_n) = C(S) \otimes M_n.$$

We equip $M_n(C(S))$ with the natural involution and C^* -norm. The canonical centre-valued trace, Γ , on $M_n(C(S))$ is defined by

$$\Gamma[a_{ij}] = \frac{1}{n} \sum_{r=1}^n a_{rr}.$$

Proposition 1.1. *Let μ be a quasi-linear functional on $M_n(C(S))$ such that $\mu(UTU^*) = \mu T$ for each unitary U and each selfadjoint T . Let $\bar{\mu}$ be defined on $C(S)$ by $\bar{\mu}(f) = \mu(f \otimes I)$. Then μ is linear and $\mu = \bar{\mu}\Gamma$.*

Proof. Let $\lambda = \bar{\mu}\Gamma - \mu$. Then λ is quasi-linear and vanishes on the centre of $M_n(C(S))$. Moreover

$$\lambda(UTU^*) = \lambda(T)$$

for each unitary U and selfadjoint T .

To simplify notation, we shall prove this proposition for $n = 2$, which is what we shall need in the next section. The proof for general n is a trivial extension of the argument.

For $x \in C(S)$ we observe that

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} x & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & x \end{pmatrix}.$$

So

$$\lambda \begin{pmatrix} x & 0 \\ 0 & 0 \end{pmatrix} = \lambda \begin{pmatrix} 0 & 0 \\ 0 & x \end{pmatrix} = \frac{1}{2} \lambda \begin{pmatrix} x & 0 \\ 0 & x \end{pmatrix} = 0.$$

Hence, for arbitrary a and b in $C(S)$,

$$\lambda \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix} = \lambda \begin{pmatrix} a & 0 \\ 0 & 0 \end{pmatrix} + \lambda \begin{pmatrix} 0 & 0 \\ 0 & b \end{pmatrix} = 0.$$

Let T be a selfadjoint element of $M_2(C(S))$. Then (see Deckard and Pearcy [10]), there exists a unitary U , such that

$$UTU^* = \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix}$$

where a and b are in $C(S)$. The $\lambda(T) = \lambda(UTU^*) = 0$. It follows that λ vanishes on $M_2(C(S))$. So $\mu = \bar{\mu}\Gamma$ and hence μ is linear.

2. LOCALLY QUASI-LINEAR TRACES

Lemma 2.1. *Let \mathcal{M} be a von Neumann algebra, and let $\mu : \mathcal{M} \rightarrow \mathbb{C}$ be locally quasi-linear and such that $\mu(uxu^*) = \mu(x)$ for each unitary u in \mathcal{M} and each selfadjoint x in \mathcal{M} . Then μ is linear.*

Proof. Let \mathcal{B} be any maximal abelian *-subalgebra of \mathcal{M} . Then \mathcal{B} is generated by its projections and so, by [3], the locally quasi-linear μ is linear on \mathcal{B} . Thus μ is quasi-linear on \mathcal{M} .

Let $\mathcal{M} = e\mathcal{M} \oplus (1 - e)\mathcal{M}$ where e is a central projection such that $e\mathcal{M}$ is of Type I_2 and $(1 - e)\mathcal{M}$ has no direct summand of Type I_2 .

For any selfadjoint x in \mathcal{M} , since e is central,

$$\mu(x) = \mu(ex) + \mu((1 - e)x).$$

By Theorem B [5] or, in the case when $\mu \geq 0$ by Christensen [9] and Yeadon [12], the restriction of μ to $(1 - e)\mathcal{M}$ is linear.

Since $e\mathcal{M}$ is of Type I_2 ,

$$e\mathcal{M} \approx M_2(C(S))$$

where S is extremally disconnected. (In fact, S is hyperstonian.) So, by Proposition 1.1, the restriction of μ to $e\mathcal{M}$ is linear. Hence μ is linear on \mathcal{M} .

Theorem 2.2. *Let $\mu : \mathcal{A} \rightarrow \mathbb{C}$ be locally quasi-linear. Suppose that either*

(i) $\mu(xy) = \mu(yx)$ for all x and all y in \mathcal{A}

or

(ii) \mathcal{A} has a unit and, for each unitary u and selfadjoint x , $\mu(uxu^*) = \mu(x)$.

Then μ is linear if, and only if, μ is uniformly continuous on \mathcal{A}_1 with respect to the weak topology.

Proof. By Lemma 1.1 [6], when μ is linear then μ is uniformly continuous on \mathcal{A}_1 with respect to the weak topology.

Let us now suppose that μ is uniformly weakly continuous on \mathcal{A}_1 .

Let H be the universal representation space of \mathcal{A} . We recall that \mathcal{A}^{**} , the second dual of \mathcal{A} , may be identified with the von Neumann envelope of \mathcal{A} in $\mathcal{L}(H)$.

Then, by the proof of Theorem 3.1 [6], μ extends uniquely to a locally quasi-linear functional $\bar{\mu} : \mathcal{A}^{**} \rightarrow \mathbb{C}$, where $\bar{\mu}$ is continuous on \mathcal{A}^{**} with respect to the strong operator topology of $\mathcal{L}(H)$.

We recall that \mathcal{A}_1 is strongly dense in the unit ball of \mathcal{A}^{**} . Hence, if (i) is satisfied, whenever x and y are in \mathcal{A}^{**} , there exist bounded nets in \mathcal{A} , (x_i) and (y_j) , where $x_i \rightarrow x$ and $y_j \rightarrow y$ is the strong operator topology of $\mathcal{L}(H)$.

$$\bar{\mu}(x_i y_j) = \mu(x_i y_j) = \mu(y_j x_i) = \bar{\mu}(y_j x_i).$$

So

$$\bar{\mu}(x y_j) = \bar{\mu}(y_j x).$$

Hence

$$\bar{\mu}(xy) = \bar{\mu}(yx).$$

Alternatively, if (ii) is satisfied, then $1 \in \mathcal{A}$ and, for each unitary u in \mathcal{A}^{**} , there is a net of unitaries (u_j) in \mathcal{A} which converges strongly to u in $\mathcal{L}(H)$. Hence $\bar{\mu}(uxu^*) = \bar{\mu}(x)$ for each selfadjoint x in \mathcal{A} . Hence $\bar{\mu}(uxu^*) = \bar{\mu}(x)$ for each selfadjoint x in \mathcal{A}^{**} .

We may now apply Lemma 2.1 to deduce that $\bar{\mu}$ is linear on \mathcal{A}^{**} and hence μ is linear on \mathcal{A} .

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