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SUPPORT CONES AND CONVEXITY OF SETS IN $\mathbb{R}^n$

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ABSTRACT. We discuss several metric characterizations of convexity of sets in non-smooth finite-dimensional Banach spaces. We describe a setting in which convexity is equivalent to the rotation-invariance of various properties, including almost convexity, radial continuity of the metric projection, and Chebyshevity. One of the tools used is a generalization of norm-smoothness which involves support cones of the unit ball.

1. INTRODUCTION

Suppose $(X, \|\cdot\|)$ is a real Banach space, K is a closed subset of X and $x \in X$. If there exists a $u \in K$ such that for every $v \in K$, $\|u-x\| \leq \|v-x\|$, then u is said to be a *best approximation* to x from K. The *metric projection* from X onto K is the set-valued mapping Π_K defined by

$$\Pi_K(x) := \{u \in K : \|u-x\| \leq \|v-x\|, v \in K\},$$

i.e., $\Pi_K(x)$ is the set of best approximations to x from K. If $\Pi_K(x)$ consists of a single point for every $x \in X$, we say that K is a *Chebyshev set.* The question of whether or not each Chebyshev set in a Hilbert space is convex is, according to Frank Deutsch [8], perhaps the most famous open problem in abstract approximation theory. A survey of the literature relating to the convexity of Chebyschev sets can be found in [8], and an exposition of some of the major theorems can be found in [10]. The study of this subject has raised several related questions about convexity characterization. The present paper is devoted to the study of convexity characterization in non-smooth, finite dimensional spaces.

Recently, L.P. Vlasov [19] introduced the concept of *almost convexity* and proved two theorems to which we will refer. We now describe Vlasov's results, beginning with the necessary definitions. If $x \in X$ and $\alpha > 0$, let $S[x,\alpha] := \{y : \|y-x\| = \alpha\}$, $B[x,\alpha] := \{y : \|y-x\| \leq \alpha\}$, and $B(x,\alpha) := \{y : \|y-x\| < \alpha\}$ be, respectively, the sphere, the closed ball, and the open ball with center x and radius α. A subset, A, of X is said to be *almost convex* if for every closed ball $B_1 := B[x,\alpha]$, which does not intersect A and every $r > 0$, there is a closed ball $B_2 := B[x',\alpha']$, such that $\alpha' > r$, and B_2 contains B_1 but does not intersect A.

Theorem 1 (Vlasov)**.** *A Chebyshev set K in a Banach space X where the metric projection Π_K is continuous, is almost convex.*

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Theorem 2 (Vlasov)**.** *In a normed linear space X, the following statements are equivalent.*

(1) *The dual space X^* is strictly convex.*
(2) *Every almost convex set is convex.*

We will assume for the remainder of this paper that the dimension of X is finite. Then there is a norm $\|\cdot\|$ on $\mathbb{R}^n$ such that $(X, \|\cdot\|)$ is isometrically isomorphic to $(\mathbb{R}^n, \|\cdot\|)$, so we will work in the latter space. Much of the theory developed so far requires that the dual space, X^*, be strictly convex. The present paper continues the exploration of metric convexity characterization (which was begun in [11], [12]) in the setting where this assumption about the dual is not made. Since $\dim X < \infty$, X^* is strictly convex if and only if X is smooth (see Corollary 2.1.1 in [9]), so our study is based in finite dimensional spaces which are not necessarily smooth. In this context, we will develop a result analogous to Theorem 2, then use the tools we have developed to arrive at other characterizations of convexity.

2. Near convexity and universal cones

We begin by considering a generalization of the definition of almost convexity. Suppose K is a closed subset of X. We propose that it be said that K is *nearly convex* if for every $x \in X \setminus K$ there is an $\alpha > 0$ such that for every $r > 0$ there is a vector $x_r \in X$ and a real number $\alpha_r > r$ with

(1) $B[x, \alpha] \subset B[x_r, \alpha_r]$,
(2) $B[x_r, \alpha_r] \cap K = \emptyset$.

We will have occasion to use the following lemma, which compares solarity and almost and near convexity. A subset K of a normed linear space X is called a *sun* if for each $x \in X$, $y \in \Pi_K(x)$, and $\lambda \geq 0$, $y \in \Pi_K(y + \lambda(x - y))$. The idea of a sun was first developed and used by Klee [13]. Suns have many of the essential approximative properties that convex sets have. For a complete discussion of suns, see [1].

Lemma 3. *Suppose $K \subset X$ is closed. Consider the following statements.*

(1) *The set K is convex.*
(2) *The set K is a sun.*
(3) *The set K is almost convex.*
(4) *The set K is nearly convex.*

Then $(1) \Rightarrow (2) \Rightarrow (3) \Rightarrow (4)$, *but no implication goes the other way.*

Proof. It is pointed out in [7] that (1) implies (2). The other implications are obvious.

To see that no implication goes the other way, first note that $\{(x, y) : y \leq \max(0, x/3)\} \subset \ell_1(2)$ is a non-convex sun. Let $K := \{(2, 0), (-2, 0)\}$. Then, clearly, K is almost convex but not a sun in $\ell_1(2)$.

Let $B_2[x_1, x_2; \alpha]$ be the closed ball centered at (x_1, x_2) with radius α in $\ell_2(2)$. Consider the "lens space" $(\mathbb{R}^2, \|\cdot\|)$ whose closed unit ball $B[0, 0; 1]$ is $B_2[-1, 0; 2] \cap B_2[1, 0; 2]$. Obviously, the ball $B[a, b; r]$ in the lens space centered at (a, b) with radius r is $B_2[a - r, b; 2r] \cap B_2[a + r, b; 2r]$.

The set K is nearly convex with respect to the lens norm. Indeed, let $x = (0, 0)$ and $\alpha = 1/\sqrt{3}$. Consider the ball $B[0, 0; \alpha]$ in the lens space. The y-intercepts of the sphere are $(0, \pm 1)$. Then by simple algebra, we easily see that, for every

$r > 1$, $B_2[r, \sqrt{3}r - 1; 2r] \supset B_2[\alpha, 0; 2\alpha]$ but $(-2, 0) \notin B_2[r, \sqrt{3}r - 1; 2r]$. Similarly, $B_2[-r, \sqrt{3}r - 1; 2r] \supset B_2[-\alpha, 0; 2\alpha]$ but $(2, 0) \notin B_2[-r, \sqrt{3}r - 1; 2r]$. Therefore, $B[0, \sqrt{3}r - 1; r] = B_2[r, \sqrt{3}r - 1; 2r] \cap B_2[-r, \sqrt{3}r - 1; 2r]$ contains neither $(-2, 0)$ nor $(2, 0)$. Clearly, $B[0, 0; \alpha] \subset B[0, \sqrt{3}r - 1, r]$. A similar argument can be used to show that, for points in $\mathbb{R}^2$ other than the origin, the conditions in the definition of nearly convex are satisfied. Hence K is nearly convex.

However, K is not almost convex with respect to the lens norm. In fact, every ball with radius 4 containing the unit ball must contain at least one of the points in K. Suppose that $B[a, b; 4] \supset B[0, 0; 1]$. It is enough to consider the case where (a, b) is in the first quadrant. Other cases can be treated similarly. In this case, we will show that $(2, 0) \in B[a, b; 4]$. It is necessary that (a, b) be located in the intersection of the first quadrant and $B_2[-4, -\sqrt{3}; 8]$. Indeed, if the center (a, b) is not in this region, then $(0, -\sqrt{3}) \in B[0, 0, ; 1] \setminus B_2[a + 4, b; 8]$; so $B[a, b; 4]$ does not contain $B[0, 0; 1]$. It is easy to check that $(2, 0) \in B[0, 3\sqrt{3}; 4]$. For a center other than $(0, 3\sqrt{3})$, a right-downward translation will guarantee that $(2, 0) \in B[a, b; 4]$. Thus, K is not almost convex. □

We will now describe a condition on $\| \cdot \|$ which guarantees the convexity of rotation-invariant nearly convex sets. By Theorem 2, the weakest possible condition that in general guarantees the convexity of almost convex sets is the smoothness of the norm. The condition we will introduce invokes a generalization of smoothness involving cones of support to the unit ball. The *norm duality* map on X is the function $J : X \to 2^{X^*}$ defined by

$$J(x) := \{x^* \in X^* : \|x^*\| = \|x\|,\ x^*(x) = \|x^*\|\|x\|\}.$$

By the Hahn-Banach Theorem, $J(x) \neq \emptyset$ for every $x \in X$. Let $x, y \in X$. The *cone of support* to $B(x, \|x - y\|)$ at y is the set

$$K(y, x) := \{z \in X : x^*(z - y) < 0 \text{ for every } x^* \in J(y - x)\}.$$

The definition of cone of support was given by Amir and Deutsch, in a general setting, in [1]. We will denote the interior of a set A by $\operatorname{int}(A)$, and the closure of A by $\overline{A}$. Given $y \in S[0, 1]$, let $\mathcal{H}$ consist of all open half-spaces which support $B(0, 1)$ at y. Then it is easy to see that $K(y, 0) = \bigcap\{H : H \in \mathcal{H}\}$. Let C_y be defined as is $K(y, 0)$, except that '$<$' is replaced by '$\leq$'. Then $C_y = \bigcap\{\overline{H} : H \in \mathcal{H}\}$, and we call C_y the support cone of $B[0, 1]$ at y. Let $\hat{C}_y = [(C_y - y) \cup (C_{-y} + y)]$. The set $\mathcal{C} := \bigcap\{\hat{C}_y : y \in S[0, 1]\}$ is a cone with vertex 0, which we call the *universal cone* of the space $(\mathbb{R}^n, \| \cdot \|)$. The universal cone was defined, in a conversation with the first author, by Wu Li.

Lemma 4. *For every $n \in \mathbb{N}$, the universal cone of $\ell_\infty(n)$ consists of all the coordinate axes and, for every $n > 2$, the universal cone of $\ell_1(n)$ is trivial.*

Proof. Note that for every vertex, y, of the unit ball, $\hat{C}_y$ consists of a closed octant, P, along with $-P$, and so contains every coordinate axis. Now we want to show that every point not in a coordinate axis is not in the universal cone. Indeed, we consider a point, say $x = (x_1, x_2, \ldots, x_n) \in \mathbb{R}^n$, which is not in a coordinate axis. We may assume that $x_1 > 0, x_2 > 0$ (other cases can be treated similarly). Let $y = (-1, 1, \ldots, 1)$, a vertex of the unit ball. It is easy to see that $\hat{C}_y = \{(x_1, x_2, \ldots, x_n) | x_1 \geq 0, x_2 \leq 0, x_3 \leq 0, \ldots, x_n \leq 0\} \cup \{(x_1, x_2, \ldots, x_n) | x_1 \leq 0, x_2 \geq 0, x_3 \leq 0, \ldots, x_n \leq 0\}$. Thus, $x \notin \hat{C}_y$.

For the second assertion note that if $x \in C_{e_j} - e_j$, then $x_j + \sum\{\alpha_i x_i : i \neq j\} \leq 0$ whenever α_i is in $\{-1, 1\}$ for each $i \neq j$. In particular, if $\alpha_i = \operatorname{sgn}(x_i)$, we obtain

$$\sum_{i \neq j} |x_i| \leq |x_j|. \tag{1}$$

A similar argument shows that (1) holds if $x \in C_{-e_j} + e_j$. If $k \neq j$, then (1) holds with j replaced by k. Adding the j and k versions of (1), we see that $\sum\{|x_i| : i \neq j, i \neq k\} \leq 0$, which implies that $x_i = 0$ whenever i is neither j nor k. Since j and k are arbitrary, it must be that $x = 0$. □

Lemma 4 shows, among other things, that the universal cone distinguishes between the two polyhedral spaces $\ell_1(n)$ and $\ell_\infty(n)$. It follows directly from the definitions that the universal cone of $(\mathbb{R}^n, \|\cdot\|)$ is $\mathbb{R}^n$ if and only if the norm is smooth. Thus, the universal cone provides a way to generalize the notion of smoothness. With this point of view, one might say that the *smoothness index* of a norm on $\mathbb{R}^n$ is the maximum dimension of subspaces contained in the universal cone. Thus, the smoothness index of a smooth norm on $\mathbb{R}^n$ is n, that of the $\ell_\infty(n)$ norm is 1, and for $n = 3$ the smoothness index of the ℓ_1 norm is 0. The "lens" norm, whose unit ball is the nonempty intersection of two distinct Euclidean spheres in $\mathbb{R}^n$, has smoothness index $(n - 1)$.

The following two results, quoted from [1] and [19] respectively, do not require that $\dim(X) < \infty$. One of the main technical results of the present paper is an extraction, in the finite dimensional setting, of the best of both.

Lemma 5 (Amir and Deutsch)**.** *If $x, y \in X$, then*

$$K(y, x) = \bigcup_{\lambda > 0} B(y + \lambda(x - y), \lambda\|x - y\|).$$

Theorem 6 (Vlasov)**.** *The following conditions for the Banach space X are equivalent.*

(1) *The dual space X^* is strictly convex.*
(2) *For any sequence of balls*

$$B_{n-1} \subset B_n = B(z_n, r_n) \qquad (n = 1, 2, \dots)$$

in X with $r_n \to \infty$, the set $\overline{\bigcup B_n}$ coincides either with X or with some half-space of X.

Theorem 6 is a specialization of Lemma 5 in the sense that Lemma 5 has no requirement similar to hypothesis (1) in Theorem 6. However, it is a generalization in the sense that its conclusion applies to a large family of chains of balls. If $\dim(X) < \infty$, compactness enables dropping the requirement that the sequence of balls be a chain. We first give a technical lemma.

Lemma 7. *If $y \in S[0, 1]$ and $k \in \mathbb{N}$, let $B_k := B(-ky, k + 1)$. Then $\operatorname{int}(C_y) = \bigcup_k B_k$.*

Proof. Note that $K(y, 0) = \bigcup_k B_k$. Since $C_y = \cap_\alpha \overline{H}_\alpha \supset \cap_\alpha H_\alpha = K(y, 0)$ and $K(y, 0)$ is an open set by Lemma 5, we have $\operatorname{int}(C_y) \supset K(y, 0)$. Conversely, for every $z \in \operatorname{int}(C_y)$, there exits a $\delta > 0$ such that $B(z, \delta) \subset C_y$. Thus for every α, $B(z, \delta) \subset \overline{H}_\alpha$, so $B(z, \delta) \subset H_\alpha$. Therefore, $B(z, \delta) \subset K(y, 0)$. Hence $z \in K(y, 0)$, so $\operatorname{int}(C_y) \subset K(y, 0)$. This completes the proof of the lemma. □

Theorem 8. *If $\{B_m := B(x_m, \alpha_m)\}$ is unbounded and $B(0,1) \subset B_m$ for every $m \in \mathbb{N}$, then there exists $y \in S[0,1]$ such that* $\operatorname{int}(C_y) \subset \bigcup_m B_m$.

Proof. For each $m \in \mathbb{N}$, let $u_m := x_m/\|x_m\|$. Since $S[0,1]$ is compact, there exist $\lambda_j \in \mathbb{N}$ and $v \in S[0,1]$ such that $v_j := u_{\lambda_j} \to v$. Let $y := -v$. Suppose $z \in \operatorname{int}(C_y)$. By Lemma 7, there is a natural number k such that $z \in B(-ky, k+1)$. Thus there is an $\epsilon_0 > 0$, such that $\|z - kv\| < k + 1 - \epsilon_0$. Since k is fixed and $\lim_{j\to\infty} v_j = v$, for every j there exists an $\epsilon_j > 0$ such that $k\|v_j - v\| < \epsilon_j$ and $\lim_{j\to\infty} \epsilon_j = 0$. Thus, for every $j \in \mathbb{N}$,

$$\begin{aligned}\|z - x_{\lambda_j}\| &\le \|z - kv\| + \|kv - kv_j\| + \|kv_j - x_{\lambda_j}\| \\ &< k + 1 - \epsilon_0 + \epsilon_j + \big|k/\|x_{\lambda_j}\| - 1\big|\|x_{\lambda_j}\| \\ &< k + 1 - \epsilon_0 + \epsilon_j + \big|\|x_{\lambda_j}\| - k\big|.\end{aligned}$$

Now there are two cases. First, suppose that there exists an $N > 0$ such that for every $j > N$, $\|x_{\lambda_j}\| < k$. Then $\|z - x_{\lambda_j}\| < 2k + 1 - \epsilon_0 + \epsilon_j - \|x_{\lambda_j}\|$. Since $\alpha_{\lambda_j} \to \infty$, there exists $M > N$ such that for every $j \ge M$, $2k + 1 - \epsilon_0 + \epsilon_j - \|x_{\lambda_j}\| < \alpha_{\lambda_j}$. Thus $\|z - x_{\lambda_j}\| < \alpha_{\lambda_j}$, i.e., $z \in B(x_{\lambda_j}, \alpha_{\lambda_j})$.

Second, suppose that for every $N > 0$ there exists some $j \ge N$ such that $\|x_{\lambda_j}\| \ge k$. Since $B(x_{\lambda_j}, \alpha_{\lambda_j}) \supset B(0,1)$, it can be seen that

$$\alpha_{\lambda_j} \ge \|x_{\lambda_j} - (-x_{\lambda_j}/\|x_{\lambda_j}\|)\| = (1 + 1/\|x_{\lambda_j}\|)\|x_{\lambda_j}\| = \|x_{\lambda_j}\| + 1.$$

So there exists $r \in \mathbb{N}$ such that

$$\|z - x_{\lambda_r}\| < 1 - \epsilon_0 + \epsilon_r + \|x_{\lambda_r}\| < \alpha_{\lambda_r}$$

where the last inequality follows from the fact that $\lim_{j\to\infty} \epsilon_j = 0$. Therefore, $z \in B(x_{\lambda_r}, \alpha_{\lambda_r})$. This concludes the proof. □

We will denote by $L(x_0, v)$ the *ray* with starting point $x_0 \in \mathbb{R}^n$ and direction $v \in \mathbb{R}^n$, i.e., the set $\{x_0 + \lambda v : \lambda \ge 0\}$.

Corollary 9. *The one-dimensional subspace V is contained in the universal cone of $(\mathbb{R}^n, \|\cdot\|)$ if and only if, for every $\alpha > 0$ and every unbounded sequence of open balls $\{B_m\}$ containing $B(0,\alpha)$, $V \cap (\cup_m B_m)$ contains a ray of the form $L(0,v)$.*

Proof. No generality is lost in assuming that $\alpha = 1$. We first claim that if $x, y \in C_y$ and $L(x,v) \subset C_y$, then $L(y,v) \subset C_y$. Indeed, for every $z \in L(y,v)$, there exists a $t_0 = t_0(z) > 0$ such that $z = y + t_0 v$. Thus, for every $\beta > 0$, if we let $r = 1 - t_0/\beta$ and notice that C_y is convex, then we have that

$$\begin{aligned}\|ry + (1-r)(x + \beta v) - z\| &= \|(1-r)(x-y) + (1-r)\beta v - t_0 v\| \\ &= t_0\|x - y\|/\beta.\end{aligned}$$

Since x, y, and t_0 are fixed, then for every $\epsilon > 0$, there exists a $\beta_0 = \beta_0(x,y,t_0) > 0$ such that for every $\beta > \beta_0, t_0\|x-y\|/\beta < \epsilon$. Since C_y is closed and z is a limit point of C_y, $z \in C_y$.

Now, we suppose V is not contained in $\mathcal{C}$. Then there exists a vector $y \in S[0,1]$ such that $(C_y - y) \cap V = \{0\}$. Thus $C_y \cap (V + y) = \{y\}$. By the above claim and $y \in C_y$, it is not the case that $C_y \cap V$ contains a ray. Hence the sequence $\{B_m\}$ described in Lemma 7 is an unbounded sequence such that $V \cap (\bigcup_m B_m)$ contains no ray.

Conversely, suppose there is an unbounded sequence $\{B_m = B(x_m, \alpha_m)\}$ containing $B(0,1)$ such that $V \cap (\bigcup_m B_m)$ contains no ray. By Theorem 8, there is a

$y \in S[0,1]$ such that $\text{int}(C_y) \subset \bigcup_m B_m$. Since $0 \in \text{int}(C_y)$, $V \cap \text{int}(C_y)$ contains no ray. Since $(C_y - y) \subset \text{int}(C_y)$, $V \cap (C_y - y)$ contains no ray. By a symmetric argument, $V \cap (C_{-y} + y)$ contains no ray. This concludes the proof. □

If $T : \mathbb{R}^n \to \mathbb{R}^n$ is a linear transformation represented by an orthogonal matrix with determinant equal to one, we will call T a *rotation*. Let $\mathcal{R}$ consist of all rotations and, given $B \subset \mathbb{R}^n$, let $\mathcal{R}(B) := \{R(B) : R \in \mathcal{R}\}$.

Theorem 10. *The following statements are equivalent.*

(1) *$(\mathbb{R}^n, \|\cdot\|)$ has nontrivial universal cone.*
(2) *For every closed subset K of $\mathbb{R}^n$, if A is nearly convex for every $A \in \mathcal{R}(K)$, then K is convex.*

Proof. Suppose that $\mathcal{C}$ contains a one-dimensional subspace L, and that K is closed but not convex. Since we may rotate and translate if necessary, no generality is lost in assuming that L contains a non-zero vector z such that the intersection of $\mathbb{R}^n \setminus K$ with the closed line segment $[z, -z]$ is the open segment $(z, -z)$. By scale-invariance, we may assume that $B[0,1] \cap K$ is empty. If K is nearly convex, then there is an $\alpha > 0$ such that for every natural number k, there is a ball $B_k := B(x_k, \alpha_k)$ containing $B(0, \alpha)$ such that $\alpha_k > k$ and $B_k \cap K$ is empty. But Corollary 9 implies that $(\bigcup_k B_k) \cap [z, -z]$ is nonempty. This contradiction shows that K is not nearly convex, so (1) implies (2).

Conversely, suppose that the universal cone is trivial. Given $u \in S[0,1]$, let $L_u := \text{span}\{u\}$. Corollary 9 implies that there exists an unbounded sequence of open balls $\{B_m^u\}$ containing $B(0,1)$ such that $L_u \cap (\bigcup_m B_m^u)$ contains no ray. Thus there exist $a_u, b_u \in \mathbb{R}$ such that

$$L_u \cap (\bigcup_m B_m^u) \subset [a_u u, b_u u].$$

Now let $a = \inf\{a_u - 1 : u \in S[0,1]\}$ and $b = \sup\{b_u + 1 : u \in S[0,1]\}$. By the compactness of $S[0,1]$, $a, b \in \mathbb{R}$. Choose $u_0 \in S[0,1]$, and let $K := \{au_0, bu_0\}$. Clearly K is not convex. Suppose $A \in \mathcal{R}(K)$. We claim that A is nearly convex. Indeed, by the construction of K, there exists $v \in S[0,1]$ such that $A = \{a'v, b'v\}$, where $a' \le a$ and $b' \ge b$. Thus there exists an unbounded sequence of open balls $\{B_m^v = B(x_m^v, \alpha_m^v)\}$ containing $B(0,1)$ such that

$$(\cup_m B_m^v) \cap A = \emptyset.$$

It is enough to consider the case that $x \in [0, b'v)$. We let

$$d_0 := \sup_m\{\|y_m\| \mid y_m = S_m^v[x_m^v, \alpha_m^v] \cap [0, bv)\}.$$

It is obvious that $d_0 \in \mathbb{R}$. Now we let $d := \|x - b'v\|$ and $\alpha := d/(2d_0)$. Clearly, $\alpha B_m^v + x \supset B(x, \alpha)$ for every $m \in \mathbb{N}$. We want to show that $b'v \notin \alpha B_m^v + x$. To see this, we consider

$$\begin{aligned}
\|b'v - \alpha x_m^v - x\| &= \left\| b'v - x - \frac{\|b'v - x\|}{2d_0} x_m^v \right\| \\
&= \alpha \left\| 2d_0 \frac{b'v - x}{\|bv - x\|} - x_m^v \right\| \\
&> \alpha \|y_m - x_m^v\| \\
&= \alpha \alpha_m^v.
\end{aligned}$$

Hence, $b'v \notin \alpha B_m^v + x$, so A is nearly convex. But, this is a contradiction. Thus, (2) implies (1). □

Theorem 10 implies that in the finite dimensional, non-smooth setting, the non-triviality of the universal cone (a property of the norm) and rotation-invariance (a property of the set K) combine to play a role in the discussion of the convexity of sets similar to the role played by the smoothness of the norm in the smooth setting.

3. Characterizations of convexity in $\mathbb{R}^n$

The tools used in the development of Theorem 10, in particular the universal cone and rotation invariance, provide a point of view from which to study various other ways of characterizing convexity. The following definition was given by Vlasov [16]. We say that K is *approximately convex* if for every $x \in X, \Pi_K(x)$ is convex. The metric projection Π_K is said to be *upper semi-continuous* if the set $\{x : \Pi_K(x) \cap C \neq \emptyset\}$ is closed for every closed set C. In [2] it was pointed out that if K is boundedly compact, then Π_K is upper semi-continuous, and the following theorem was proven.

Theorem 11 (Blatter, Morris and Wulbert). *Let X be a reflexive Banach space. Let K be a proximinal subset of X with metric projection Π_K such that for every x in X, $\Pi_K(x)$ is compact and convex. Then if Π_K is weakly upper semi-continuous, K is a sun.*

Note that in the finite-dimensional setting, weak continuity is equivalent to continuity. We now have the tools we need to prove a theorem characterizing convexity in spaces with non-trivial universal cone.

Theorem 12. *Suppose that $(\mathbb{R}^n, \|\cdot\|)$ has non-trivial universal cone and K is a closed subset of $\mathbb{R}^n$. Consider the following statements.*

(1) *For every $A \in \mathcal{R}(K)$, A is Chebyshev.*
(2) *The set K is convex.*
(3) *For every $A \in \mathcal{R}(K)$, A is a sun.*
(4) *For every $A \in \mathcal{R}(K)$, A is almost convex.*
(5) *For every $A \in \mathcal{R}(K)$, A is nearly convex.*
(6) *For every $A \in \mathcal{R}(K)$, A is approximately convex.*

Then (1) *implies* (2)*, and statements* (2) *through* (6) *are equivalent.*

Proof. Suppose (1) holds. By Corollary 2 on page 237 in [10], Π_A is continuous. Hence by Theorem 1, A is almost convex, so Lemma 3 implies that A is nearly convex. Then Theorem 10 implies that K is convex.

It follows from Lemma 3 that (2) $\Rightarrow$ (3) $\Rightarrow$ (4) $\Rightarrow$ (5). And, (5) $\Rightarrow$ (2) follows from Theorem 10.

If K is convex and $A \in \mathcal{R}(K)$, then A is convex, and an easy calculation shows that (2) $\Rightarrow$ (6). We conclude the proof by showing that (6) $\Rightarrow$ (2). Indeed, let $A \in \mathcal{R}(K)$ be given. If for every $x \in \mathbb{R}^n, \Pi_A(x)$ is convex, then it is easy to see that the conditions in Theorem 11 are satisfied in the present finite-dimensional setting. Thus A is a sun. It follows from (3) that K is convex. □

A normed linear space E is smooth if and only if every sun in E is convex. The proof of this fact appeared in a paper by Vlasov, a translation of which was published in 1961 [15]. Vlasov attributes the result to Efimov and Stechkin. See also [3]. Theorem 12 gives a non-smooth analogue of this result in the finite-dimensional setting. It also gives a non-smooth finite-dimensional analogue of the following theorem of Vlasov [16]: *In a smooth Banach space, every boundedly compact approximately convex set is convex.* It is not the case that (2) implies (1) in Theorem 12. Indeed, let $K := \{(x, y) : x = 0\} \subset \ell_\infty(2)$.

We will now consider the relationship between convexity of sets in $\ell_\infty(n)$ and radial continuity of the metric projection. The Hausdorff metric on compact sets of $\mathbb{R}^n$ is defined by

$$H(A, B) := \max\{d(A, B), d(B, A)\}$$

where $d(C, D) := \sup_{c \in C} d(c, D)$. We say Π_K is *continuous* at x if $\lim_{z \to x} H(\Pi_K(x), \Pi_K(z)) = 0$. If K is a closed subset of $\ell_\infty(n)$, we say that Π_K is *radially continuous* at $x \in \mathbb{R}^n \backslash K$ if, for every $y \in \Pi_K(x)$, the restriction of Π_K to the ray $\{x + \lambda(x - y) : \lambda \geq 0\}$ is continuous at x, and we say that Π_K is radially continuous if it is radially continuous at x for every $x \in \mathbb{R}^n \backslash K$. In the mid 70's, Brosowski and Deutsch defined various types of radial continuity and showed how these are related to convexity [4], [5], [6]. The separate properties they defined were interesting in their own right and characterized certain phenomena. The following theorem, apparently due to Vlasov, was cited in [8].

Theorem 13 (Vlasov)**.** *Suppose that K is a closed subset of a Hilbert space. The following statements are equivalent.*

(1) *The metric projection Π_K is radially continuous.*
(2) *The set K is convex.*

Now we will consider subsets of $\ell_\infty(n)$ and obtain a theorem analogous to Theorem 13. First, some definitions. The metric projection, Π_K, is said to be *lower semicontinuous* if the set $\{v : \Pi_K(v) \cap U \neq \emptyset\}$ is open for every open set U. It is sometimes convenient to use the following equivalent definition of local lower semicontinuity. The metric projection, Π_K, is lower semicontinuous at x if for every $z \in \Pi_K(x)$, for every sequence $\{x_m\} \to x$, and for every $\epsilon > 0$ there exists a natural number N such that for all $m \geq N$ there exists $y_m \in \Pi_K(x_m)$ such that $\|y_m - z\| < \epsilon$. Let $E := \{e_1, \ldots, e_n\}$ be the canonical basis of $\mathbb{R}^n$.

Theorem 14. *Suppose that K is a closed subset of $\ell_\infty(n)$. Then the following statements are equivalent.*

(1) *The set K is convex.*
(2) *For every $A \in \mathcal{R}(K)$, the metric projection Π_A is radially continuous.*

Proof. We assume without loss of generality that K is compact. If $u \in \mathbb{R}^n$ and $1 \leq i \leq n$, we denote the ith coordinate of u by $u(i)$.

Suppose that K is convex but there exists $A \in \mathcal{R}(K)$ and $x \in \mathbb{R}^n \backslash A$ such that Π_A is not radially continuous at x. Since continuity of the metric projection is equivalent to lower semicontinuity of the metric projection in the finite-dimensional setting, there exists $y \in \Pi_K(x)$ such that Π_K restricted to $\{x + \lambda(x - y) : \lambda \geq 0\}$ is not lower semicontinuous at x, i.e., there exists a sequence $\{\lambda_m\}$, which decreases to 0, a vector $z \in \Pi_K(x)$ and a positive real number ϵ such that, with $x_m := x + \lambda_m(x - y)$, we have $d(z, \Pi_K(x_m)) \geq \epsilon$ for every natural number m. We also have

$$\begin{aligned}\|x_m - z\| &\leq \|x_m - x\| + \|x - z\| \\ &= \|x_m - x\| + \|x - y\| \\ &= \|x_m - y\| \\ &= d(x_m, K)\end{aligned}$$

where the last identity is true because K is a sun. Thus $z \in \Pi_K(x_m)$, a contradiction.

Conversely, suppose K is not convex. Then, rotating and rescaling if necessary, there exists $\beta > 0$ such that, with $z = \beta e_n$, we have $[z, -z] \cap K = \{z, -z\}$ and $B(0,1) \cap K = \emptyset$. Since K is compact, there exists $\gamma > 0$ such that $B(-\eta e_n, 1) \cap K$ is empty for $0 \leq \eta < \gamma$ but nonempty for $\eta > \gamma$. Choose $w \in \Pi_K(-\gamma e_n)$.

By Lemma 4, span$\{e_n\}$ is contained in the universal cone of $\ell_\infty(n)$. By the choice of w and Corollary 9, $z \in \bigcup_{m>1} B_m$, where $B_m := B(y_m, r_m)$, with $y_m = w + m(-\gamma e_n - w)$ and $r_m = \|y_m - w\| = m\|\gamma e_n + w\|$. Thus there is an $m_0 > 1$ such that $B_m \cap K$ is empty for $1 < m < m_0$ but nonempty for $m > m_0$. Let $x := y_{m_0}$. We will show that Π_K is not radially continuous at x.

Let F_w be the union of all of the faces of $B[x, r_{m_0}]$ which contain w. If $b \in F_w$ and $m > m_0$, let σ satisfy the equation $x = \sigma y_m + (1-\sigma)w$. Then $\sigma \in (0,1)$. If $\hat{b} := \sigma b + (1-\sigma)w$, then $\hat{b} \in F_w$, so $\|\hat{b} - x\| = \|w - x\|$ and it follows that $\sigma\|b - y_m\| = \sigma\|w - y_m\|$. Hence $b \in B[y_m, r_m]$. It follows that for every $m > m_0$

$$F_w \imath S[y_m, r_m]. \tag{3}$$

By the definition of m_0, for every $m > m_0$,

$$d(y_m, K) < r_m. \tag{4}$$

Suppose that $m_j \downarrow m_0$ and there exist $h_j \in \Pi_K(y_{m_j})$ such that $h_j \to w$. Since no h_j is in $B(x, r_{m_0})$, there exists an $N \in \mathbb{N}$ such that for every $j \geq N$ the line segment $[y_{m_j}, h_j]$ must intersect F_w. By (3), $\|y_{m_j} - h_j\| \geq r_{m_j}$, which contradicts (4). Thus Π_K is not radially continuous at x. □

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