PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 124, Number 8, August 1996

ON RELATIVE HAUSDORFF MEASURES OF NONCOMPACTNESS AND RELATIVE CHEBYSHEV RADII IN BANACH SPACES

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(Communicated by Palle E. T. Jorgensen)

Abstract. In this paper we prove some formulae and evaluations on relative Hausdorff measures of noncompactness and relative Chebyshev radii in various Banach spaces. We generalize the Lifschitz constant $\kappa(X)$ and introduce a function $\tilde{\kappa}_X(\cdot)$.

1. Introduction

In this paper X will always denote a real Banach space and A will be a bounded subset of X. The classical Hausdorff measure of noncompactness $\chi(A)$ is defined as the infimum of numbers $\varepsilon > 0$ such that A can be covered with a finite number of balls of radii smaller than ε. The absolute Chebyshev radius $r(A)$ is defined as the infimum of numbers $\varepsilon > 0$ such that A can be covered with a ball of a radius ε. Thus we have

$$r(A) = \inf_{y \in X} \sup_{x \in A} \|x - y\|.$$

The concept of relative Chebyshev centers and radii is a natural generalization of the notion of the absolute Chebyshev center and radius. In particular, relative Chebyshev centers have been well studied in recent years (see for instance [13]). For a given set $\emptyset \neq G \subset X$ the relative radius $r_G(A)$ is given by

$$r_G(A) = \inf_{y \in G} \sup_{x \in A} \|x - y\|.$$

Similarly we can define the relative Hausdorff measure of noncompactness $\chi_G(A)$ as the infimum of those $r > 0$ such that A can be covered with a finite number of balls with centers in G of radii smaller than r. If $G = A$ we have the so-called inner Hausdorff measure of noncompactness. If G is a linear subspace of X and $A \subset G$, we obtain the (classical) Hausdorff measure of noncompactness in a subspace G [16].

Section 2 contains formulae on $\chi_G(A)$ and $r_G(A)$ in terms of the Hausdorff distance.

Received by the editors September 19, 1994 and, in revised form, February 24, 1995.

1991 *Mathematics Subject Classification.* Primary 41A65, 46B20, 47H09; Secondary 41A50, 47H10.

Key words and phrases. Chebyshev radius, Hausdorff measure of noncompactness, Hausdorff distance, Lifschitz constant, L^p spaces, space of continuous functions.

In section 3 we generalize the Lifschitz constant $\kappa(X)$ and give evaluations of $\chi_G(A)$ and $r_G(A)$ in any Banach spaces.

In section 4 a function $\tilde{\kappa}_X(\cdot)$ is defined. With the help of this function we give some stronger evaluations of $\chi_G(A)$ and $r_G(A)$ than those given in section 3.

Section 5 presents some applications of previous ideas to Hilbert spaces, L^p spaces and some spaces with the norm "supremum". We generalize the formula proved by Smith and Ward in [15].

2. General remarks

Let $B(x,r)$ denote the closed ball centered at $x \in X$ with radius $r > 0$, and let $\operatorname{dist}(x,A)$ denote the distance to a point x from $A \subset X$. We shall also use the notation:

$$B(A,r) = \{x \in X : \operatorname{dist}(x,A) \le r\},$$

$$\operatorname{dist}(A,G) = \inf_{x\in A}\inf_{y\in G}\|x-y\|, \qquad A, G \subset X.$$

Denote by $\mathcal{M}$ the family of all nonempty bounded subsets of X. For $A, G \in \mathcal{M}$ put

$$d(A,G) = \inf\{r > 0 : A \subset B(G,r)\},$$

$$D(A,G) = \max\{d(A,G), d(G,A)\}$$

and call them the nonsymmetric and symmetric Hausdorff distance between A and G, respectively. Sometimes we shall use the symbol $d(A,G)$ with unbounded G. It is well known that D is a metric defined on the family of all bounded and closed subsets of X. We shall also use the following symbols:

$\mathcal{N}$–the family of all relatively compact and nonempty subsets of X,

$\mathcal{N}^0$–the family of all nonempty finite subsets of X,

$\mathcal{N}^s$–the family of all subsets of X consisting of exactly one element.

If $\mathcal{Z}$ is a family of subsets of X, then we shall write

$$d(A,\mathcal{Z}) = \inf_{G\in\mathcal{Z}} d(A,G) \quad\text{and}\quad D(A,\mathcal{Z}) = \inf_{G\in\mathcal{Z}} D(A,G).$$

Using this notation it is easily seen that

$$\chi(A) = D(A,\mathcal{N}) = d(A,\mathcal{N}) = D(A,\mathcal{N}^0) = d(A,\mathcal{N}^0)$$

and

$$r(A) = D(A,\mathcal{N}^s) = d(A,\mathcal{N}^s).$$

Proposition 2.1. *Let $\emptyset \ne G \subset X$. Then*

$$\chi_G(A) = \inf_{F\in\mathcal{N}}[d(A,F)+d(F,G)] = \inf_{F\in\mathcal{N}^0}[d(A,F)+d(F,G)],$$

$$r_G(A) = \inf_{F\in\mathcal{N}^s}[d(A,F)+d(F,G)].$$

Proof. The equality $\inf_{F\in\mathcal{N}}[d(A,F)+d(F,G)] = \inf_{F\in\mathcal{N}^0}[d(A,F)+d(F,G)]$ follows from the fact that for every $\varepsilon > 0$ and $F \in \mathcal{N}$ there exists $F_0 \in \mathcal{N}^0$ such that $d(F,F_0) < \varepsilon$ and $d(F_0,F) < \varepsilon$. Therefore it is sufficient to prove that $\chi_G(A) = \inf_{F\in\mathcal{N}^0}[d(A,F)+d(F,G)]$. Fix $\varepsilon > 0$ and $F \in \mathcal{N}^0$, and choose $x \in A$. By the definition of $d(A,F)$ and $d(F,G)$ there exist $f \in F$ and $g \in G$ such that $\|x-f\| \le d(A,F)+\frac{\varepsilon}{2}$ and $\|f-g\| \le d(F,G)+\frac{\varepsilon}{2}$. Hence

$$\|x-g\| \le \|x-f\|+\|f-g\| \le d(A,F)+d(F,G)+\varepsilon$$

and so

$$\chi_G(A) \le d(A,F)+d(F,G)$$

for every $F \in \mathcal{N}^0$. This implies

$$\chi_G(A) \le \inf_{F\in\mathcal{N}^0}[d(A,F)+d(F,G)].$$

We have

$$\begin{aligned}\chi_G(A) &\le \inf_{F\in\mathcal{N}^0}[d(A,F)+d(F,G)] \le \inf_{\substack{F\in\mathcal{N}^0\\ F\subset G}}[d(A,F)+d(F,G)]\\ &= \inf_{\substack{F\in\mathcal{N}^0\\ F\subset G}} d(A,F) = \chi_G(A),\end{aligned}$$

which completes the proof. Similar considerations apply to $r_G(A)$. □

Write

$$D_1(A,B) = d(A,B)+d(B,A).$$

Notice that $D_1(A,B)$ is a metric defined on the family of all bounded and closed subsets of X. From Proposition 2.1 we obtain

Proposition 2.2. *Let $A \subset X$ be a bounded, nonempty set. Then*

$$\begin{aligned}\chi_A(A) &= D_1(A,\mathcal{N}) = D_1(A,\mathcal{N}^0),\\ r_A(A) &= D_1(A,\mathcal{N}^s). \qquad \square\end{aligned} \tag{1}$$

3. EVALUATIONS ON $\chi_G(A)$ AND $r_G(A)$ WITH THE USE OF THE FUNCTION $\kappa_X(\cdot)$

For $\varepsilon \ge 0$ write

$$\mathcal{H}^\varepsilon(A) = \{F \in \mathcal{N}^0 : A \subset B(F,(1+\varepsilon)\chi(A))\},$$

$$E^\varepsilon(A) = \{y \in X : A \subset B(y,(1+\varepsilon)r(A))\}.$$

Note that for $\varepsilon > 0$ $\mathcal{H}^\varepsilon(A) \ne \emptyset$ and $E^\varepsilon(A) \ne \emptyset$ for every bounded $A \subset X$. Proposition 2.1 now gives

Proposition 3.1. *Let $\emptyset \ne G \subset X$. Then*

$$\chi_G(A) \le \chi(A) + \lim_{\varepsilon\to 0^+} d(\mathcal{H}^\varepsilon(A),G),$$

$$r_G(A) \le r(A) + \lim_{\varepsilon\to 0^+} \operatorname{dist}(E^\varepsilon(A),G). \qquad \square$$

To find the converse evaluations on $\chi_G(A)$ and $r_G(A)$ let us recall the definition of the Lifschitz constant $\kappa(X)$ of a Banach space X :

$$\begin{aligned}\kappa(X) = \sup\Bigg\{ k>0 : &\bigvee_{0<\mu,\alpha<1}\ \bigwedge_{x,y\in X}\ \bigwedge_{r>0} \|x-y\| \ge (1-\mu)r\\ &\Rightarrow \bigvee_{z\in X} B(x,(1+\mu)r)\cap B(y,k(1+\mu)r) \subset B(z,\alpha r)\Bigg\}.\end{aligned}$$

This constant plays an important role in fixed point theorems for uniformly Lipschitzian mappings. We will need to generalize it.

Definition 3.2. Let X be a Banach space. $\kappa_X(\cdot)$ is a function defined on $(0,+\infty)$ by

$$\kappa_X(d) = \sup\left\{k > 0 : \bigvee_{0<\mu,\alpha<1} \bigwedge_{x,y\in X} \bigwedge_{r>0} \|x-y\| \geq (1-\mu)rd \right.$$
$$\left. \Rightarrow \bigvee_{z\in X} B(x,(1+\mu)r) \cap B(y,k(1+\mu)r) \subset B(z,\alpha r)\right\}.$$

Theorem 3.3. *Let X be a Banach space, $A \subset X$ be a bounded set and $\emptyset \neq G \subset X$. Then*

$$\chi_G(A) \geq \chi(A)\kappa_X\left(\frac{d_1}{\chi(A)}\right) \quad \text{if } \chi(A) \neq 0, \tag{2}$$

$$r_G(A) \geq r(A)\kappa_X\left(\frac{d_2}{r(A)}\right) \quad \text{if } r(A) \neq 0, \tag{3}$$

where

$$d_1 = \lim_{\varepsilon\to 0^+} \sup_{F\in\mathcal{H}^\varepsilon(A)} \operatorname{dist}(F,G), \qquad d_2 = \lim_{\varepsilon\to 0^+} d(E^\varepsilon(A),G).$$

Proof. Write $\chi_G(A) = k$, $\chi(A) = r$. Assume that $\frac{k}{r} < \kappa_X(\frac{d_1}{r})$. Then we can find $0 < \mu$, $\alpha < 1$ such that for every $x, y \in X$ with $\|x-y\| \geq (1-\mu)d_1$ there exists $z \in X$ satisfying

$$B(x,(1+\mu)r) \cap B(y,k(1+\mu)) \subset B(z,\alpha r).$$

Choose $g_1, g_2, \ldots, g_n \in G$ satisfying

$$A \subset \bigcup_{i=1}^n B(g_i,(1+\mu)k)$$

and $F = \{x_1, x_2, \ldots, x_m\} \in \mathcal{H}^\mu(A)$ such that

$$\bigwedge_{\substack{i=1,\ldots,n\\ j=1,\ldots,m}} \|g_i - x_j\| \geq \operatorname{dist}(F,G) \geq (1-\mu)d_1.$$

Obviously $A \subset \bigcup_{j=1}^m B(x_j,(1+\mu)r)$. But

$$B(g_i,(1+\mu)k) \cap B(x_j,(1+\mu)r) \subset B(z_{ij},\alpha r)$$

for some $z_{ij} \in X$ and we obtain

$$\begin{aligned} A &\subset \bigcup_{i=1}^n B(g_i,(1+\mu)k) \cap \bigcup_{j=1}^m B(x_j,(1+\mu)r) \\ &= \bigcup_{i=1}^n \bigcup_{j=1}^m B(g_i,(1+\mu)k) \cap B(x_j,(1+\mu)r) \subset \bigcup_{i=1}^n \bigcup_{j=1}^m B(z_{ij},\alpha r), \end{aligned}$$

which contradicts $\chi(A) = r$. Thus (2) is proved. The proof for (3) is similar. $\square$

It is now natural to consider: When is $\kappa_X(d) > 1$? First recall that the modulus of convexity of a Banach space X is the function $\delta_X : [0,2] \to [0,1]$ defined by

$$\delta_X(\varepsilon) = \inf\left\{1 - \frac{\|x+y\|}{2} : \|x\| \le 1, \|y\| \le 1, \|x-y\| \ge \varepsilon\right\}$$

and

$$\varepsilon_0(X) = \sup\{\varepsilon \in [0,2] : \delta_X(\varepsilon) = 0\}.$$

We follow the ideas of Downing and Turett [3] (see also [8]).

Proposition 3.4. *In any Banach space X :*

$$\kappa_X(d) > 1 \Leftrightarrow \varepsilon_0(X) < d.$$

Proof. Let $\varepsilon_0(X) < d$ and let $u \in B(x,(1+\mu)r) \cap B(y,(1+\mu)kr)$, where $\mu > 0$, $k > 1$ and $\|x-y\| \ge (1-\mu)dr$. Then $\|u-x\| \le (1+\mu)r$, $\|u-y\| \le (1+\mu)rk$ and taking $z = \frac{x+y}{2}$ we obtain

$$\|u-z\| \le \left(1 - \delta_X\left(\frac{(1-\mu)d}{(1+\mu)k}\right)\right)(1+\mu)kr \le \alpha r$$

with $\alpha < 1$ if μ is sufficiently small and k is sufficiently close to 1. Hence $\kappa_X(d) > 1$ (see also [8]). To prove the converse assume that $\kappa_X(d) > 1$ and $\varepsilon_0(X) \ge d$. Fix $\varepsilon > 0$. There exist $x, y \in X$ such that $\|x\| = \|y\| = 1$, $\|x-y\| = d$ and $\|\frac{x+y}{2}\| \ge 1-\varepsilon$. For any $\mu > 0$ the segment

$$[0, x+y] \subset B(x,(1+\mu)) \cap B(y,(1+\mu)).$$

But $\|x+y\| \ge 2-2\varepsilon$ and ε is arbitrary so $\kappa_X(d) = 1$, which is a contradiction. $\square$

Notice that for every $0 < d < \frac{4}{\varepsilon_0(X)}$ there exists exactly one number $b \ge 1$ such that $b(1-\delta_X(\frac{d}{b})) = 1$ (it follows from the fact that δ_X is an increasing, continuous function on $[0,2)$ and $\lim_{\varepsilon\to 2^-} \delta_X(\varepsilon) = 1 - \frac{\varepsilon_0(X)}{2}$ [7]). Denote by $b_X(\cdot)$ a function depending on the argument $d < \frac{4}{\varepsilon_0(X)}$ satisfying

$$b_X(d)\left(1 - \delta_X\left(\frac{d}{b_X(d)}\right)\right) = 1.$$

Similar considerations to that given above lead us to

Proposition 3.5. *In any Banach space X :*

$$\kappa_X(d) \ge b_X(d) \quad \text{for } d < \frac{4}{\varepsilon_0(X)}. \qquad \square$$

The constant $b_X(1)$ was introduced by Goebel and Kirk in [6].

4. Evaluations on $\chi_G(A)$ and $r_G(A)$ with the use of the function $\tilde{\kappa}_X(\cdot)$

In section 3 the expressions d_1, d_2 in Theorem 3.3 are not natural. Now we find stronger and more useful evaluations on $\chi_G(A)$ and $r_G(A)$. We start with the following definition:

Definition 4.1. Let X be a Banach space. $\tilde{\kappa}_X(\cdot)$ is a function defined on $(0, +\infty)$ by

$$\tilde{\kappa}_X(d) = \sup \left\{ k > 0 : \bigvee_{0<\alpha<1} \bigwedge_{x,y \in X} \bigwedge_{r>0} \bigvee_{z \in X} \|z - y\| \leq \alpha d r \wedge B(x, r) \cap B(y, kr) \subset B(z, r) \right\}.$$

Theorem 4.2. *Let X be a Banach space, $A \subset X$ be a bounded set and $\emptyset \neq G \subset X$. Then*

$$\chi_G(A) \geq \chi(A) \tilde{\kappa}_X \left(\frac{d_\chi}{\chi(A)} \right) \quad \textit{if } \chi(A) \neq 0, \tag{4}$$

$$r_G(A) \geq r(A) \tilde{\kappa}_X \left(\frac{d_r}{r(A)} \right) \quad \textit{if } r(A) \neq 0, \tag{5}$$

where

$$d_\chi = \lim_{\varepsilon \to 0^+} d(\mathcal{H}^\varepsilon(A), G), \qquad d_r = \lim_{\varepsilon \to 0^+} \operatorname{dist}(E^\varepsilon(A), G).$$

Proof. Write $\chi_G(A) = k$, $\chi(A) = r$. Assume that $\frac{k}{r} < \tilde{\kappa}_X(\frac{d_\chi}{\chi(A)})$. Fix $\varepsilon > 0$ and take $0 < \delta \leq \varepsilon$ such that $d(\mathcal{H}^\delta(A), G) \geq d_\chi(1 - \varepsilon)$. Then we can find $\alpha < 1$ such that for every $x, y \in X$ there exists $z \in X$ satisfying

$$\|z - y\| \leq \alpha d_\chi (1 + \delta) \leq \alpha d_\chi (1 + \varepsilon)$$

and

$$B(x, (1 + \delta) r) \cap B(y, (1 + \delta) k) \subset B(z, (1 + \delta) r).$$

Choose $g_1, g_2, \ldots, g_n \in G$ such that

$$A \subset \bigcup_{i=1}^n B(g_i, (1 + \delta) k)$$

and $F = \{x_1, x_2, \ldots, x_m\} \in \mathcal{H}^\delta(A)$. Then

$$B(g_i, (1 + \delta) k) \cap B(x_j, (1 + \delta) r) \subset B(z_{ij}, (1 + \delta) r)$$

for some $z_{ij} \in X$ satisfying

$$\|z_{ij} - g_i\| \leq \alpha d_\chi (1 + \varepsilon), \qquad i = 1, \ldots, n, \ j = 1, \ldots, m. \tag{6}$$

Hence

$$\begin{aligned} A &\subset \bigcup_{i=1}^n B(g_i, (1 + \delta) k) \cap \bigcup_{j=1}^m B(x_j, (1 + \delta) r) \\ &= \bigcup_{i=1}^m \bigcup_{j=1}^m B(g_i, (1 + \delta) k) \cap B(x_j, (1 + \delta) r) \subset \bigcup_{i=1}^n \bigcup_{j=1}^m B(z_{ij}, (1 + \delta) r) \end{aligned}$$

and $\{z_{ij}\} \in \mathcal{H}^\delta(A)$. From the definition of $d(\mathcal{H}^\delta(A), G)$ there exist i_0, j_0 such that

$$\|z_{i_0 j_0} - g_{i_0}\| \geq d(\mathcal{H}^\delta(A), G) \geq d_\chi (1 - \varepsilon),$$

which contradicts (6) if ε is sufficiently small. Thus (4) is proved. The proof for (5) is similar. □

Let $l_{x,y}$ denote the line $\{\alpha x+\beta y : \alpha, \beta \in \mathcal{R}, \alpha+\beta=1\}$ containing $x, y \in X$, and let $[x, y]$ denote the segment $\{\alpha x+\beta y : \alpha, \beta \geq 0, \alpha+\beta=1\}$. In the next section we will need the following

Lemma 4.3. *In any Banach space X and for every $d > 0$ we have*

$$\tilde{\kappa}_X(d) \geq \tilde{\kappa}_X^{lin}(d),$$

where

$$\tilde{\kappa}_X^{lin}(d) = \sup \left\{ k > 0 : \bigvee_{0<\alpha<1} \bigwedge_{x,y \in X} \bigwedge_{r>0} \|x-y\| \leq rd \right.$$
$$\left. \Rightarrow \bigvee_{z \in l_{x,y}} \|z-y\| \leq \alpha rd \wedge B(x,r) \cap B(y,kr) \subset B(z,r) \right\}.$$

Proof. Obviously $d-1 \leq \tilde{\kappa}_X^{lin}(d) \leq d+1$. It is sufficient to consider $\tilde{\kappa}_X^{lin}(d) > d-1$. Assume that $\tilde{\kappa}_X(d) < \tilde{\kappa}_X^{lin}(d)$ for some $d > 0$. Then there exist $\alpha < 1, x, y \in X$, $r > 0$ satisfying $\|x-y\| > dr$ and $x_1, z \in [x, y]$ such that $\|z-y\| \leq \alpha rd$, $\|x_1-y\| = dr$ and

$$B(x,r) \cap B(y,kr) \not\subset B(z,r) \supset B(x_1,r) \cap B(y,kr).$$

That means there exists $g \in X$ satisfying $\|g-x\| < r$, $\|g-y\| \leq kr$ and $\|g-x_1\| > r$. Let $y_1 \in l_{x,y}$ and $\|y-y_1\| = kr$. Then $\|x_1-y_1\| < r$ and we have $\|x_1-g_1\| = r$ for some $g_1 \in [y_1, g]$. Moreover $\|g_1-y\| \leq kr$ and thus $\|g_1-z\| \leq r$. But $\|g_1-x\| < r$ (since $\|x-y_1\| \leq r$ and $\|x-g\| < r$). This contradicts $\|x_1-g_1\| = r$. □

5. Applications

In this section we use the results of section 4. First observe that (in particular) in reflexive spaces $E^0(A) \neq \emptyset$ for every bounded set A (see [5], [9] for more details).

Let H be a Hilbert space. It is not difficult to verify that $\kappa_H(d) = \sqrt{1+d^2}$ and $\tilde{\kappa}_H(d) = \sqrt{1+d^2}$. Combining Proposition 3.1 and Theorem 4.2 we get

Corollary 5.1. *Let H be a Hilbert space, $A \subset H$ be a bounded set and $\emptyset \neq G \subset H$. Then*

$$\lim_{\varepsilon \to 0^+} \sqrt{[\chi(A)]^2 + [d(\mathcal{H}^\varepsilon(A), G)]^2} \leq \chi_G(A) \leq \chi(A) + \lim_{\varepsilon \to 0^+} d(\mathcal{H}^\varepsilon(A), G),$$

$$\sqrt{[r(A)]^2 + [\mathrm{dist}(E^0(A), G)]^2} \leq r_G(A) \leq r(A) + \mathrm{dist}(E^0(A), G). \qquad \square$$

Let $X = L^p$ or l^p, $p > 2$. We will adapt methods contained in [11], where it is proved that

$$\|tx+(1-t)y\|^p + t^{\frac{p}{2}}(1-t)^{\frac{p}{2}}\|x-y\|^p \leq t\|x\|^p + (1-t)\|y\|^p$$

for $x, y \in X, 0 \leq t \leq 1$. Take $\|x\| \leq r(1+\mu)$, $\|y\| \leq rk(1+\mu)$ and $\|x-y\| \geq (1-\mu)dr$. We have

$$\|tx+(1-t)y\| \leq (t(1+\mu)^p + (1-t)(1+\mu)^p k^p - t^{\frac{p}{2}}(1-t)^{\frac{p}{2}}(1-\mu)^p d^p)^{\frac{1}{p}} r.$$

If we find $t_0 \in (0,1)$ such that

$$t_0 + (1-t_0)k^p - t_0^{\frac{p}{2}}(1-t_0)^{\frac{p}{2}} d^p < 1, \tag{7}$$

then $k < \kappa_X(d)$. Moreover $\|t_0x + (1-t_0)y - y\| = t_0\|x-y\| \leq t_0dr$ if $\|x-y\| \leq dr$. Hence there would be also $k < \tilde{\kappa}_X^{lin}(d)$. But (7) is equivalent to

$$k < \left(1 + t_0^{\frac{p}{2}}(1-t_0)^{\frac{p}{2}-1}d^p\right)^{\frac{1}{p}}.$$

Putting $t_0 = \frac{p}{2(p-1)}$ we have $k < (1 + \frac{d^p}{2^{p-1}}p^{\frac{p}{2}}(p-1)^{1-p}(p-2)^{\frac{p}{2}-1})^{\frac{1}{p}}$. Hence (see also [11])

$$\kappa_X(d) \geq \left(1 + \frac{d^p}{2^{p-1}}p^{\frac{p}{2}}(p-1)^{1-p}(p-2)^{\frac{p}{2}-1}\right)^{\frac{1}{p}} > \left(1 + \frac{d^p}{2^{p-1}}\right)^{\frac{1}{p}}$$

and

$$\tilde{\kappa}_X^{lin}(d) \geq \left(1 + \frac{d^p}{2^{p-1}}p^{\frac{p}{2}}(p-1)^{1-p}(p-2)^{\frac{p}{2}-1}\right)^{\frac{1}{p}} > \left(1 + \frac{d^p}{2^{p-1}}\right)^{\frac{1}{p}}.$$

Proposition 3.1, Theorem 4.2 and Lemma 4.3 yield

Corollary 5.2. *Let $X = L^p$ or l^p, $p > 2$, $A \subset X$ be a bounded set and $\emptyset \neq G \subset X$. Then*

$$\begin{aligned}\lim_{\varepsilon\to 0^+} &\left([\chi(A)]^p + \frac{[d(\mathcal{H}^\varepsilon(A), G)]^p}{2^{p-1}}p^{\frac{p}{2}}(p-1)^{1-p}(p-2)^{\frac{p}{2}-1}\right)^{\frac{1}{p}} \\ &\leq \chi_G(A) \leq \chi(A) + \lim_{\varepsilon\to 0^+} d(\mathcal{H}^\varepsilon(A), G),\end{aligned}$$

$$\begin{aligned}&\left([r(A)]^p + \frac{[\mathrm{dist}(E^0(A), G)]^p}{2^{p-1}}p^{\frac{p}{2}}(p-1)^{1-p}(p-2)^{\frac{p}{2}-1}\right)^{\frac{1}{p}} \\ &\leq r_G(A) \leq r(A) + \mathrm{dist}(E^0(A), G). \qquad \square\end{aligned}$$

Let $X = L^p$ or l^p, $1 < p \leq 2$. Lim, Xu, and Xu [12] and Smarzewski [14] proved that

$$\|tx + (1-t)y\|^2 + (p-1)t(1-t)\|x-y\|^2 \leq t\|x\|^2 + (1-t)\|y\|^2$$

for $x, y \in X$ and $0 < t < 1$. Similar considerations to those given above give $\kappa_X(d) \geq \sqrt{1 + (p-1)d^2}$ and $\tilde{\kappa}_X^{lin}(d) \geq \sqrt{1 + (p-1)d^2}$. Hence we get

Corollary 5.3. *Let $X = L^p$ or l^p, $1 < p \leq 2$, $A \subset X$ be a bounded set and $\emptyset \neq G \subset X$. Then*

$$\lim_{\varepsilon\to 0^+} \sqrt{[\chi(A)]^2 + (p-1)[d(\mathcal{H}^\varepsilon(A), G)]^2} \leq \chi_G(A) \leq \chi(A) + \lim_{\varepsilon\to 0^+} d(\mathcal{H}^\varepsilon(A), G),$$

$$\sqrt{[r(A)]^2 + (p-1)[\mathrm{dist}(E^0(A), G)]^2} \leq r_G(A) \leq r(A) + \mathrm{dist}(E^0(A), G). \qquad \square$$

Let $X = C([0,1], \mathcal{R})$ be the space of real, continuous functions defined on $[0,1]$. We have $\kappa_X(d) = \max\{1, d-1\}$ so Theorem 3.3 is not valid in this case. However we may use the function $\tilde{\kappa}_X(\cdot)$. We formulate our theorem in a more general case:

Theorem 5.4. *Let $M(T, \mathcal{R})$ be the space of bounded, real functions defined on T with the norm $\|x\| = \sup_{t\in T}|x(t)|$. Let X be a linear subspace of $M(T, \mathcal{R})$ containing the function $x(t) \equiv 1$ and closed with respect to taking maximum and minimum, i.e.:*

$$\bigwedge_{x,y\in X} z(t) = \max\{x(t), y(t)\} \in X, \qquad \bigwedge_{x,y\in X} z(t) = \min\{x(t), y(t)\} \in X.$$

If $A \subset X$ is bounded and $\emptyset \neq G \subset X$, then

$$\chi_G(A) = \chi(A) + \lim_{\varepsilon \to 0^+} d(\mathcal{H}^\varepsilon(A), G), \tag{8}$$

$$r_G(A) = r(A) + \lim_{\varepsilon \to 0^+} \operatorname{dist}(E^\varepsilon(A), G). \tag{9}$$

Proof. It is enough to show that $\tilde{\kappa}_X(d) = 1 + d$. Let $x, y \in X$, $0 < \alpha < 1$, $r > 0$, $k < 1 + d$ and $u \in B(x, r) \cap B(y, kr)$. Write:

$$z(t) = \begin{cases} y(t) + (k-1)r & \text{if } x(t) \geq y(t) + (k-1)r, \\ x(t) & \text{if } y(t) - (k-1)r < x(t) < y(t) + (k-1)r, \\ y(t) - (k-1)r & \text{if } x(t) \leq y(t) - (k-1)r. \end{cases}$$

Then $\|z - y\| \leq (k-1)r \leq \alpha dr$ if α is sufficiently close to 1. Moreover $\|u - z\| \leq r$ so $u \in B(z, r)$ and thus $\tilde{\kappa}_X(d) \geq 1 + d$. The inequality $\tilde{\kappa}_X(d) \leq 1 + d$ is obvious. □

Remark 5.5. In particular Theorem 5.4 is valid for $C([0,1], \mathcal{R}), c, c_0, l^\infty$.

Remark 5.6. The formula (9) in the case of continuous functions was first proved by Smith and Ward in [15] and then by Franchetti and Cheney in [4].

Remark 5.7. The case $G = \{x \in X : \int_{[0,1]} x \, d\mu = 0\}$ if $X = C([0,1], \mathcal{R})$, where μ is a real, normalized Borel measure on $[0,1]$, was studied in [16]:

$$\chi_G(A) = \max\{\chi(A), \omega^+(A), \omega^-(A)\},$$

where

$$\chi(A) = \frac{1}{2} \lim_{h \to 0^+} \sup_{x \in A} \sup\{|x(t) - x(s)| : |t - s| \leq h, \ t, s \in [0,1]\},$$

$$\omega^+(A) = \lim_{h \to 0^+} \sup_{x \in A} \left(\int_{[0,1]} R_h^+ x \, d\mu_1 - \int_{[0,1]} R_h^- x \, d\mu_2 \right),$$

$$\omega^-(A) = \lim_{h \to 0^+} \sup_{x \in A} \left(\int_{[0,1]} R_h^+ x \, d\mu_2 - \int_{[0,1]} R_h^- x \, d\mu_1 \right),$$

$$(R_h^+ x)(t) = \sup\{x(t + \theta h) : -1 \leq \theta \leq 1, t + \theta h \in [0,1]\},$$

$$(R_h^- x)(t) = \inf\{x(t + \theta h) : -1 \leq \theta \leq 1, t + \theta h \in [0,1]\},$$

$\mu = \mu_1 - \mu_2$ is the Jordan decomposition of μ.

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