

A SUBMARTINGALE INEQUALITY

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ABSTRACT. This paper extends Burkholder's inequality between a nonnegative submartingale and a process strongly differentially subordinate to it.

1. STATEMENT OF THE INEQUALITY

Let $0 \leq \alpha \leq 1$ and $1 < p < \infty$. Suppose that $f = (f_n)_{n \geq 0}$ and $g = (g_n)_{n \geq 0}$ are adapted to a filtration $(\mathcal{F}_n)_{n \geq 0}$ of a probability space (Ω, \mathcal{F}, P) . Here f is a nonnegative submartingale and g is \mathbb{R}^ν -valued, where ν is a positive integer. With $f_n = d_0 + \cdots + d_n$ and $g_n = e_0 + \cdots + e_n$ ($n \geq 0$) we assume that

$$(1.1) \quad |e_n| \leq |d_n| \quad (n \geq 0),$$

$$(1.2) \quad |\mathbb{E}(e_n | \mathcal{F}_{n-1})| \leq \alpha \mathbb{E}(d_n | \mathcal{F}_{n-1}) \quad (n \geq 1).$$

Then, with $\|f\|_p = \sup_{n \geq 0} \|f_n\|_p$, we have the inequality

$$(1.3) \quad \|g\|_p \leq (r-1)\|f\|_p$$

where $r = \max\{(\alpha+1)p, p/(p-1)\}$ and r is best possible.

Remarks. 1. The case $\alpha = 1$ is shown in Burkholder's paper [3].

2. For martingales f and g that satisfy (1.1) and have values in a Hilbert space, the inequality (1.3) is shown in Burkholder's paper [2]. There $r = \max\{p, p/(p-1)\}$.

3. *Best possible* means that if $r-1$ in (1.3) is replaced by a strictly smaller number β , then the opposite inequality $\|g\|_p > \beta\|f\|_p$ holds for some probability space and some f and g as above.

2. OUTLINE OF THE PROOF OF THE INEQUALITY

We may assume $\|f\|_p < \infty$. Also, we may assume

$$(2.1) \quad f_{n-1} > 0 \quad \text{and} \quad |g_{n-1} + te_n| > 0 \quad \text{for all } t \in \mathbb{R} \text{ and } n \geq 1.$$

Indeed, for each $0 < \varepsilon < 1$, the processes f^ε and g^ε , where $f_n^\varepsilon = f_n + \varepsilon$ and $g^\varepsilon = (g_n, \varepsilon)$, satisfy (2.1) and all the assumptions in Section 1. Here g^ε is a process in $\mathbb{R}^{\nu+1}$. Since $\|g\|_p \leq \|g^\varepsilon\|_p$ and $\|f^\varepsilon\|_p \leq \|f\|_p + \varepsilon$, the inequality (1.3) follows from $\|g^\varepsilon\|_p \leq (r-1)\|f^\varepsilon\|_p$ as ε tends to 0.

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Put $S = (0, \infty) \times \mathbb{R}^\nu$ and define functions U and V on S by

$$(2.2) \quad U(x, y) = (|y| - (r-1)x)(x + |y|)^{p-1},$$

$$(2.3) \quad V(x, y) = |y|^p - ((r-1)x)^p.$$

Then the inequality (1.3) follows from

$$(2.4) \quad \mathbb{E}V(f_n, g_n) \leq 0 \quad \text{for all } n \geq 0$$

which is a consequence of the inequalities:

$$(2.5) \quad \mathbb{E}V(f_n, g_n) \leq p(1 - 1/r)^{p-1} \mathbb{E}U(f_n, g_n) \quad \text{for all } n \geq 0,$$

$$(2.6) \quad \mathbb{E}U(f_n, g_n) \leq \mathbb{E}U(f_{n-1}, g_{n-1}) \quad \text{for all } n \geq 1,$$

$$(2.7) \quad \mathbb{E}U(f_0, g_0) \leq 0.$$

3. PROOF OF THE INEQUALITY

It remains to prove the inequalities (2.5), (2.6) and (2.7).

Proof of (2.5). Notice that (2.5) is proved if $V(x, y) \leq p(1 - 1/r)^{p-1}U(x, y)$ for all $(x, y) \in S$, and that the latter inequality holds if and only if it holds with x replaced by $x/(x + |y|)$ and y replaced by $y/(x + |y|)$. Thus, it is enough to prove the inequality for $x \in (0, 1]$ and $|y| = 1 - x$, or, equivalently, to prove $F \leq 0$ on $[0, 1]$ where

$$(3.1) \quad F(x) = (1 - x)^p - [(r-1)x]^p - p(1 - rx)(1 - 1/r)^{p-1}.$$

If $0 < x < 1$, then

$$(3.2) \quad F'(x) = -p[(1 - x)^{p-1} + (r-1)^p x^{p-1}] + rp(1 - 1/r)^{p-1},$$

$$(3.3) \quad F''(x) = p(p-1)[(1 - x)^{p-2} - (r-1)^p x^{p-2}].$$

Notice that $0 < 1/r < 1$ and $F(1/r) = F'(1/r) = 0$.

If $p = 2$, then $F''(x) = 2[1 - (r-1)^2] \leq 0$ on $(0, 1)$ because $r \geq (\alpha + 1)2 \geq 2$. Thus F has the maximum at $x = 1/r$ which proves $F \leq 0$ on $[0, 1]$ in this case.

Now assume $1 < p < 2$. The strict concavity of $\log x$ implies that

$$(3.4) \quad \log 1 > a \log a + (1 - a) \log(a + 1) \quad \text{or} \quad (a + 1)^{a-1} > a^a \quad \text{for } 0 < a < 1,$$

so $a = p - 1$ gives $(p - 1)^{p-1} < p^{p-2}$. Hence $r^{p-1} \geq (p/(p-1))^{p-1} > p$. Thus $F(1) = -(r-1)^p - p(1-r)(1-1/r)^{p-1} = (p - r^{p-1})(r-1)^p/r^{p-1} < 0$. Let x^* be the zero of F'' . Computation gives $x^*[1 + (r-1)^{p/(p-2)}] = 1, 0 < 1/r < x^* < 1$, and $F''(x) < 0$ if $0 < x < x^*$. Thus the maximum of F on $[0, x^*]$ is $F(1/r) = 0$. Also, $F''(x) > 0$ if $x^* < x < 1$. Thus F is convex on $[x^*, 1]$, hence $F \leq 0$ on $[x^*, 1]$; because $F(x^*) \leq 0$ and $F(1) < 0$.

The case $p > 2$ is handled similarly. Here $F(0) < 0$ follows from the inequality

$$(3.5) \quad (a + 1)^{a-1} < a^a \quad \text{for } a > 1$$

which follows from (3.4) upon replacing a by $1/a$.

Proof of (2.6). The proof depends on the inequality: If (x, y) and $(x + h, y + k)$ belong to $(0, \infty) \times \mathbb{R}^\nu$ with $|h| \geq |k|$, and $y + tk \neq 0$ for all $t \in \mathbb{R}$, then

$$(3.6) \quad U(x + h, y + k) \leq U(x, y) + \varphi(x, y)h + \psi(x, y) \cdot k$$

where $\varphi(x, y) = U_x(x, y)$ and $\psi(x, y) = U_y(x, y)$, that is,

$$\begin{aligned} \varphi(x, y) &= [(p - r)(x + |y|) - (p - 1)rx](x + |y|)^{p-2}, \\ \psi(x, y) &= [p(x + |y|) - (p - 1)rx](x + |y|)^{p-2}y/|y|. \end{aligned}$$

For a proof of (3.6) in the case $\alpha = 1$, see [3]; the case $0 \leq \alpha < 1$ requires only the further observation that $r \geq p/(p - 1)$ and $r \geq p$.

Now (2.1) and (3.6) give

$$(3.7) \quad U(f_n, g_n) \leq U(f_{n-1}, g_{n-1}) + \varphi(f_{n-1}, g_{n-1})d_n + \psi(f_{n-1}, g_{n-1}) \cdot e_n$$

where all the random variables are integrable: for example the last one is integrable because of the assumption (1.1) and the estimate $|\psi(x, y)| \leq pr(x + |y|)^{p-1}$. Conditioning on \mathcal{F}_{n-1} and using the assumption (1.2), one sees from (3.7) that

$$\begin{aligned} \mathbb{E}U(f_n, g_n) - \mathbb{E}U(f_{n-1}, g_{n-1}) \\ \leq \mathbb{E}([\varphi(f_{n-1}, g_{n-1}) + \alpha|\psi(f_{n-1}, g_{n-1})|] \mathbb{E}(d_n | \mathcal{F}_{n-1})). \end{aligned}$$

Here f is a submartingale, hence $\mathbb{E}(d_n | \mathcal{F}_{n-1}) \geq 0$. Thus the inequality (2.6) follows from the inequality $\varphi(x, y) + \alpha|\psi(x, y)| \leq 0$ which, by homogeneity, is equivalent to the inequality $G(x) \leq 0$ if $0 \leq x \leq 1$, where

$$G(x) = (p - r) - (p - 1)rx + \alpha|p - (p - 1)rx|.$$

Clearly G is convex, hence it suffices to show $G(0) \leq 0$ and $G(1) \leq 0$. Here $G(0) = (\alpha + 1)p - r \leq 0$, and noting that $p - (p - 1)r \leq 0$, one has

$$\begin{aligned} G(1) &= (p - r) - (p - 1)r - \alpha[p - (p - 1)r] \\ &= -r + (1 - \alpha)[p - (p - 1)r] \leq 0. \end{aligned}$$

This proves (2.6).

Proof of (2.7). Since $r \geq 2$ we have $|y| - (r - 1)x \leq |y| - x \leq 0$ if $|y| \leq x$. Hence (2.7) follows from (2.2), the assumption (1.1) which gives $|g_0| \leq |f_0|$ when $n = 0$, and the nonnegativity of f_0 .

4. SHARPNESS OF THE INEQUALITY

Case 1. Suppose that $1 < p \leq (\alpha + 2)/(\alpha + 1)$. Then $r = p/(p - 1)$ and so $r - 1$ is the best constant in (1.3) since it is already the best possible constant if f is a nonnegative martingale as can be seen in (5.90) and (5.91) of [1].

Case 2. Here $p > (\alpha + 2)/(\alpha + 1)$ so $r = (\alpha + 1)p$. Choose a small $\eta > 0$ so that $\eta(p - 1)(\alpha + 1) < 2$. Define $(x_n)_{n \geq 1}$ and $(\pi_n)_{n \geq 1}$ by

$$rx_n = 2 + n(\alpha + 1)\eta \quad \text{and} \quad \pi_1 = \frac{1}{2}, \pi_{n+1} = \frac{1}{2} \prod_{k=1}^n \left(1 - \frac{\eta}{x_k}\right).$$

Now we define a filtration $(\mathcal{F}_n)_{n \geq 0}$ on the Lebesgue probability space $[0, 1)$ as follows: $\mathcal{F}_0 = \{\emptyset, [0, 1)\}$ and for $n \geq 1$, $\mathcal{F}_{2n-1} = \mathcal{F}_{2n}$ is generated by the partition of $[0, 1)$ determined by $0 < \pi_n < \pi_{n-1} < \dots < \pi_1 < 1$.

Using the same notation for the interval $[a, b]$ and its characteristic function on $[0, 1]$ we put

$$\begin{aligned} d_0 &= e_0 = [0, 1), & d_1 &= -e_1 = -[0, \frac{1}{2}) + [\frac{1}{2}, 1), \\ d_{2n} &= \eta[0, \pi_n), & e_{2n} &= \alpha d_{2n}, \\ d_{2n+1} &= -e_{2n+1} = -\eta[0, \pi_{n+1}) + (x_n - \eta)[\pi_{n+1}, \pi_n). \end{aligned}$$

Let $f_n = d_0 + \dots + d_n$ and $g_n = e_0 + \dots + e_n$. Then one checks that f and g are adapted to (\mathcal{F}_n) , that f is a nonnegative submartingale, and that (1.1) and (1.2) are satisfied. Also,

$$\begin{aligned} f_{2n+1} &= \sum_{1 \leq k \leq n} x_k [\pi_{k+1}, \pi_k) + 2 [\frac{1}{2}, 1), \\ g_{2n+1} &= rx_n [0, \pi_{n+1}) + (r - 1) \sum_{k=1}^n x_k [\pi_{k+1}, \pi_k). \end{aligned}$$

Thus we get

$$\|g_{2n+1}\|_p \geq (r - 1) \|f_{2n+1}\|_p \left(\frac{A_n}{A_n + 2^p} \right)^{1/p}$$

with $A_n = a_1 + \dots + a_n$ where $a_n = |x_n|^p (\pi_n - \pi_{n+1})$.

Notice that each pair of stopped processes, f^{2n+1} and $g^{2n+1} = (g_{k \wedge (2n+1)})_{k \geq 0}$, satisfies all the assumptions of Section 1. Thus, in order to establish sharpness in Case 2, it is enough to show that $\lim_{n \rightarrow \infty} A_n = \infty$. Put $b = (\alpha + 1)\eta$. By Taylor's formula, as $n \rightarrow \infty$,

$$\begin{aligned} \frac{a_{n+1}}{a_n} &= \left(1 + \frac{b}{2 + nb} \right)^{p-1} \left(1 - \frac{r\eta}{2 + nb} \right) \\ &= \left(1 + \frac{(p-1)b}{2 + bn} + O\left(\left(\frac{b}{2 + nb} \right)^2 \right) \right) \left(1 - \frac{r\eta}{2 + nb} \right) \\ &= \left(1 + \frac{p-1}{n} + O\left(\frac{1}{n^2} \right) \right) \left(1 - \frac{p}{n} + O\left(\frac{1}{n^2} \right) \right) \\ &= 1 - \frac{1}{n} + O\left(\frac{1}{n^2} \right). \end{aligned}$$

Now recall the Gauss test for the convergence of a series of positive numbers:

if $a_n > 0$ and $\frac{a_{n+1}}{a_n} = 1 - \frac{\lambda}{n} + O\left(\frac{1}{n^2}\right)$, then $\sum a_n < \infty$ if and only if $\lambda > 1$.

In our case, $\lambda = 1$ so $\sum a_n = \lim A_n = \infty$.

This completes the proof that $r - 1$ is the best constant in the inequality (1.3).

Remarks. 1. The inequality (1.3) holds even for $\alpha > 1$. The proof in Section 3 works provided that $\alpha \leq [1 + \sqrt{1 + (p-1)^2}]/(p-1)$. For $\alpha > 1$ sharpness of r could not be established.

2. If $0 < \|f\|_p < \infty$, then the inequality (1.3) is strict unless $\alpha = 0$ and $p = 2$: this follows as in [3] because $r > 2$ unless $\alpha = 0$ and $p = 2$.

3. The inequality (1.3), or, more precisely, the function U in (2.2), leads to analogous inequalities for Itô processes and smooth functions on Euclidean domains. These follow as in [3]. For example, the condition $|\Delta v| \leq |\Delta u|$ in [3] is replaced

by $|\Delta v| \leq \alpha |\Delta u|$. The inequality $\|v\|_p \leq (p^{**} - 1)\|u\|_p$ in [3] becomes $\|v\|_p \leq (r - 1)\|u\|_p$.

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