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FIXED POINTS OF CONTRACTIVE MULTIVALUED MAPS

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ABSTRACT. For a class of contractive multivalued maps defined on a complete absolute retract and with closed bounded values, the set of fixed points is proved to be an absolute retract. This result unifies and extends to arbitrary absolute retracts both Theorem 1 by Ricceri [Atti Accad. Naz. Lincei Rend. Cl. Sci. Fis. Mat. Natur. (8) **81** (1987), 283–286] and Theorem 1 by Bressan, Cellina, and Fryszkowski [Proc. Amer. Math. Soc. **112** (1991), 413–418].

INTRODUCTION

Let X be a complete metric space and let φ be a contractive multivalued map from X into itself, with nonempty values. If $\varphi(x)$ is closed for every $x \in X$, then the set $Fix(\varphi)$ of all fixed points of φ is known to be nonempty [7, Corollary 3]. Since, contrary to the singlevalued case, $Fix(\varphi)$ is not necessarily a singleton, it makes sense to look for topological properties of it.

In this framework, some years ago, B. Ricceri established the following result (see [15, Théorème 1]).

Theorem A. *Let E be a Banach space, let X be a nonempty, convex, closed subset of E, and let φ be a contractive multivalued map from X into itself, with convex closed values. Then the set $Fix(\varphi)$ is an absolute retract.*

Later on, several papers were devoted to possible extensions and applications of Theorem A [5, 10, 11, 13, 14, 16, 17]. In particular, when $X = L^1(T)$ for some measure space T, A. Bressan, A. Cellina, and A. Fryszkowski obtained the following result (see [5, Theorem 1]).

Theorem B. *Suppose φ is a contractive multivalued map from $L^1(T)$ into itself, with bounded, closed, decomposable values. Then the conclusion of Theorem A holds.*

In the present paper we try to establish a result which unifies and extends to arbitrary absolute retracts both Theorem A and Theorem B. To this end, we first define, for a lower semicontinuous multivalued map φ from X into itself, the notion of selection property with respect to a given family $\mathcal{D}$ of metric spaces

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(Definition 2.1). Next, we prove that if X is a complete absolute retract, φ is a contractive multivalued map having the selection property with respect to the family of all metric spaces, and $\varphi(x)$ is closed and bounded for every $x \in X$, then the set $Fix(\varphi)$ is an absolute retract (Theorem 2.3). This result has a variety of interesting special cases (see Theorems 2.4–2.7). As an example, Theorem 2.6 gives Theorem A for φ with bounded values, while Theorem B is a consequence of Theorem 2.7.

We emphasize that in Theorem A no boundedness condition on the values of φ is assumed. So, the problem to unify Theorems A and B for multivalued maps with possibly unbounded values is still open.

1. Basic definitions and preliminary results

Let A, B be two sets. The set-theoretic difference between A and B is written $A \setminus B$; their symmetric difference is $A \Delta B = (A \setminus B) \cup (B \setminus A)$. The symbol $\sharp A$ stands for the cardinality of the set A, and χ_A is the characteristic function of A.

Let (X, d) be a metric space and let A be a subset of X. We denote by $\mathrm{cl}(A)$ the closure of A. If A is nonempty, $x \in X$, and $r > 0$, we set

$$d(x, A) = \inf_{z \in A} d(x, z), \qquad O_r(A) = \{z \in X : d(z, A) < r\},$$
$$B(x, r) = \{z \in X : d(x, z) < r\}.$$

For every pair of nonempty sets $A, B \subseteq X$, we define

$$d_H(A, B) = \inf\{r > 0 : A \subseteq O_r(B), B \subseteq O_r(A)\}.$$

Throughout this paper, we write $\mathcal{M}$ to denote the family of all metric spaces. If $X \in \mathcal{M}$ and $h : X \to \mathbb{R}$, the symbol $\operatorname{supp} h$ stands for $\mathrm{cl}(\{x \in X : h(x) \neq 0\})$.

Let $X \in \mathcal{M}$ and let A be a nonempty subset of X. We say that A is a retract of X if there exists a continuous function $r : X \to A$ such that $r(x) = x$ for all $x \in A$. The space X is called an absolute retract for metric spaces (briefly, $X \in AR(\mathcal{M})$) if, for any $Y \in \mathcal{M}$ and any nonempty closed set $Y_0 \subseteq Y$, every continuous function $f_0 : Y_0 \to X$ has a continuous extension $f : Y \to X$ over Y.

The family $AR(\mathcal{M})$ is rather large. As an example, retracts of absolute retracts are also absolute retracts and any convex subset of a normed space is an absolute retract (see [4]).

(1.1) **Proposition.** *Let X be a separable metric space and let A be a nonempty closed subset of X. If $X \in AR(\mathcal{M})$ and, for any separable space $Y \in \mathcal{M}$ and any nonempty closed set $Y_0 \subseteq Y$, every continuous function $f_0 : Y_0 \to A$ admits a continuous extension over Y, then $A \in AR(\mathcal{M})$.*

Proof. The assumptions yield a continuous function $r : X \to A$ such that $r(x) = x$ for all $x \in A$. Let $Y \in \mathcal{M}$, let Y_0 be a nonempty closed subset of Y, and let $f_0 : Y_0 \to A$ be a continuous function. Since $X \in AR(\mathcal{M})$, there is a continuous function $g : Y \to X$ fulfilling $g(y) = f_0(y)$ for every $y \in Y_0$. The function $f = r \circ g$ is, obviously, a continuous extension of f_0. □

Let $X \in \mathcal{M}$ and let n be a nonnegative integer. We say that X is n-connected if every function $f_0 : S^n \to X$ has a continuous extension over K^{n+1}, where S^n and K^{n+1} denote the unit sphere and the closed unit ball of $\mathbb{R}^{n+1}$, respectively. The space X is called infinitely connected when it is n-connected for each $n \geq 0$. We say that X is contractible if there exist a continuous function $h : X \times [0, 1] \to X$

and a point $x_0 \in X$ such that $h(x,0) = x, h(x,1) = x_0$ for all $x \in X$. Evidently, any contractible space is infinitely connected [4, p. 30].

Let $(T, \mathcal{F}, \mu)$ be a finite, positive, nonatomic measure space and let $(E, \|\cdot\|)$ be a Banach space. We denote by $L^1(T,E)$ the Banach space of all (equivalence classes of) μ-measurable functions $u : T \to E$ such that the function $t \to \|u(t)\|$ is μ-integrable, equipped with the norm

$$\|u\|_{L^1(T,E)} = \int_T \|u(t)\|\, d\mu.$$

Throughout this paper, we assume that the space $L^1(T,E)$ is separable.

According to [6, 8], a nonempty set $K \subseteq L^1(T,E)$ is said to be decomposable if, for every $u_1, u_2 \in K$ and every μ-measurable subset A of T, one has

$$\chi_A \cdot u_1 + (1 - \chi_A) \cdot u_2 \in K.$$

Some basic facts about decomposable sets in $L^1(T,E)$ are collected in the following

(1.2) *Remarks.* (1.2.1) A simple computation shows that the open (or closed) unit ball of $L^1(T,E)$ is not decomposable.

(1.2.2) Owing to Lemma 4 of [9], it is easily seen that every decomposable subset of $L^1(T,E)$ is contractible and, consequently, infinitely connected.

(1.2.3) From [5, Theorem 1] it follows that any bounded, closed, decomposable subset of $L^1(T,E)$ is an absolute retract.

Let $X, Y \in \mathcal{M}$. A multivalued map φ from X into Y (in symbols, $\varphi : X \to Y$) is a function from X into the family of all nonempty, closed, bounded subsets of Y. If $X \subseteq Y$, we write $Fix(\varphi)$ for $\{x \in X : x \in \varphi(x)\}$. For every set $W \subseteq Y$, we define $\varphi^{-1}(W) = \{x \in X : \varphi(x) \subseteq W\}$. If, for any closed (open) set $W \subseteq Y$, the set $\varphi^{-1}(W)$ is closed (open) in X, we say that φ is lower (upper) semicontinuous. The multivalued map φ is called continuous when it is both lower and upper semicontinuous. Let d_1 be the metric of X and let d_2 be the metric of Y. We say that φ is Lipschitzean if there exists a real number $L \geq 0$ such that $d_{2H}(\varphi(x'), \varphi(x'')) \leq L d_1(x', x'')$ for all $x', x'' \in X$. When $L < 1$, φ is called contractive. It is a simple matter to see that any Lipschitzean multivalued map is lower semicontinuous. Finally, a function $f : X \to Y$ satisfying $f(x) \in \varphi(x)$ for every $x \in X$ is said to be a selection of φ.

The most famous continuous selection theorem is the following result by E. Michael (see Example 1.3* and Theorem 3.2″ in [12] or [15, Theorem 2]).

(1.3) **Theorem.** *Let $X \in \mathcal{M}$, let E be a Banach space, and let $\varphi : X \to E$ be a lower semicontinuous multivalued map with convex values. Then, for any nonempty closed set $X_0 \subseteq X$, every continuous selection f_0 from $\varphi|_{X_0}$ admits a continuous extension f over X such that $f(x) \in \varphi(x)$ for all $x \in X$.*

The preceding result is no longer true without assuming φ convex-valued (see [1, p. 68]), so that the problem to find continuous selections from multivalued maps with nonconvex values has been investigated by numerous authors. One of the possible answers is performed in [2], where the concept of simplicial convexity is introduced. We conclude this section with a more abstract approach to the above-mentioned problem, which is essential for our purposes.

(1.4) **Definition.** Let $X \in \mathcal{M}$ and let $M(X)$ be a family of closed subsets of X, with the following properties:

(1.4.1) $X \in M(X)$ and, if $\{A_i\}_{i\in I}$ is a subfamily of $M(X)$, then $\bigcap_{i\in I} A_i \in M(X)$.
(1.4.2) For every $k \in \mathbb{N}$ and every $x_1, x_2, \ldots, x_k \in X$, the set $A(x_1, x_2, \ldots, x_k) = \bigcap\{A : A \in M(X),\ x_1, x_2, \ldots, x_k \in A\}$ is infinitely connected.
(1.4.3) $\{x\} \in M(X)$ for each $x \in X$.
(1.4.4) To every $\varepsilon > 0$ there corresponds $\delta > 0$ such that, for any $A \in M(X)$, any $k \in \mathbb{N}$, and any $x_1, x_2, \ldots, x_k \in O_\delta(A)$, one has $A(x_1, x_2, \ldots, x_k) \subseteq O_\varepsilon(A)$.
(1.4.5) $\mathrm{cl}(A \cap B(x, r)) \in M(X)$ for all $A \in M(X)$, $x \in X$, and $r > 0$.

Then we say that $M(X)$ is a Michael family of subsets of X.

This notion is closely related to the existence of continuous selections. Indeed, a simple argument based on Proposition 1.5 and Theorem 3.5 of [2] produces the following

(1.5) **Proposition.** *Let $X, Y \in \mathcal{M}$ and let $\varphi : X \to Y$ be a lower semicontinuous multivalued map. If Y is complete and there exists a Michael family $M(Y)$ of subsets of Y such that $\varphi(x) \in M(Y)$ for all $x \in X$, then the conclusion of Theorem 1.3 holds.*

The preceding result gains in interest if we realize that significant classes of sets are examples of Michael families.

(1.6) **Examples.** (1.6.1) Let X be a convex subset of a normed space and let $M(X)$ be the class of all sets $A \subseteq X$ such that $A = \emptyset$ or A is convex and closed in X. Then $M(X)$ is a Michael family of subsets of X.

(1.6.2) Let $X \in \mathcal{M}$, and let $M(X)$ be a simplicial convexity on X (in the sense of [2, Definition 1.3]), whose elements are closed in X. Then $M(X)$ is a Michael family of subsets of X (see [2]).

2. Fixed points of contractive multivalued maps

The aim of this section is to establish a result which, in the setting of contractive multivalued maps with bounded values, unifies and extends to arbitrary absolute retracts both Theorem 1 by B. Ricceri [15] and Theorem 1 by A. Bressan, A. Cellina, and A. Fryszkowski [5].

(2.1) **Definition.** Let $X \in \mathcal{M}$, let φ be a lower semicontinuous multivalued map from X into itself, and let $\mathcal{D} \subseteq \mathcal{M}$. We say that φ has the *selection property with respect to* $\mathcal{D}$ when, for any $Y \in \mathcal{D}$, any pair of continuous functions $f : Y \to X$ and $h : Y \to]0, +\infty[$ such that

$$\psi(y) = \mathrm{cl}(\varphi(f(y)) \cap B(f(y), h(y))) \neq \emptyset, \qquad y \in Y,$$

and any nonempty closed set $Y_0 \subseteq Y$, every continuous selection g_0 from $\psi|_{Y_0}$ admits a continuous extension g over Y fulfilling $g(y) \in \psi(y)$ for all $y \in Y$. If $\mathcal{D} = \mathcal{M}$, then we say that φ has the *selection property* (in symbols, $\varphi \in SP(X)$).

To emphasize some interesting features of the above concept, we make the following

(2.2) *Remarks.* (2.2.1) It is well known that the intersection of lower semicontinuous multivalued maps is not necessarily lower semicontinuous. Nevertheless, a simple argument, based on Theorems 1.3.6 and 1.3.9 of [3], shows that the map ψ introduced in Definition 2.1 is lower semicontinuous. Consequently, owing to Theorem 1.3, for any $\mathcal{D} \subseteq \mathcal{M}$ there exist multivalued maps having the selection property with respect to $\mathcal{D}$.

(2.2.2) From [1, Theorems 6 and 7, p. 53] it follows that every continuous and compact-valued or Lipschitzean multivalued map $\varphi \in SP(X)$ satisfies the conclusion of Theorem 1.3.

(2.2.3) Let $X \in \mathcal{M}$ and let $\varphi : X \to X$ be a lower semicontinuous multivalued map. If X is complete and there exists a Michael family $M(X)$ of subsets of X such that $\varphi(x) \in M(X)$ for all $x \in X$, then $\varphi \in SP(X)$. This is an immediate consequence of Proposition 1.5.

(2.2.4) Let X be a nonempty closed subset of $L^1(T, E)$ and let $\varphi : X \to X$ be a lower semicontinuous multivalued map, with decomposable values. Then φ has the selection property with respect to the family of all separable metric spaces (see Theorem 3.1).

We are in a position now to establish the main result of this paper.

(2.3) **Theorem.** *Let X be a complete absolute retract and let $\varphi : X \to X$ be a contractive multivalued map. Suppose $\varphi \in SP(X)$. Then the set $Fix(\varphi)$ is a complete absolute retract.*

Proof. By Corollary 3 of [7], the set $Fix(\varphi)$ is nonempty, and a simple argument ensures that it is also closed. Therefore, to perform the proof, it is sufficient to show that if $Y \in \mathcal{M}$, Y^* is a nonempty closed subset of Y, and $f^* : Y^* \to Fix(\varphi)$ is a continuous function, then there exists a continuous extension $f : Y \to Fix(\varphi)$ of f^* over Y.

Let d be the metric of X, let $L \in]0, 1[$ be such that $d_H(\varphi(x'), \varphi(x'')) \leq Ld(x', x'')$ for all $x', x'' \in X$, and let $M \in]1, L^{-1}[$. The assumption $X \in AR(\mathcal{M})$ yields a continuous function $f_0 : Y \to X$ fulfilling $f_0(y) = f^*(y)$ in Y^*. We claim that there is a sequence $\{f_n\}$ of continuous functions from Y into X, with the following properties:

(i) $f_n|_{Y^*} = f^*$ for every $n \in \mathbb{N}$.
(ii) $f_n(y) \in \varphi(f_{n-1}(y))$ for all $y \in Y, n \in \mathbb{N}$.
(iii) $d(f_n(y), f_{n-1}(y)) \leq L^{n-1}d(f_1(y), f_0(y)) + M^{1-n}$ for every $y \in Y, n \in \mathbb{N}$.

To see this, we proceed by induction on n. From [1, Theorem 7, p. 53] it follows that the function $h_0 : Y \to]0, +\infty[$ defined by

$$h_0(y) = \sup\{d(f_0(y), z) : z \in \varphi(f_0(y))\} + 1, \qquad y \in Y,$$

is continuous; moreover, one has $\varphi(f_0(y)) \cap B(f_0(y), h_0(y)) = \varphi(f_0(y))$ for all $y \in Y$. Bearing in mind that $\varphi \in SP(X)$, we obtain a continuous function $f_1 : Y \to X$ satisfying $f_1(y) = f^*(y)$ in Y^* and $f_1(y) \in \varphi(f_0(y))$ in Y. Hence, conditions (i), (ii), and (iii) are true for f_1. Suppose now we have constructed p continuous functions $f_1, f_2, \dots, f_p$ from Y into X in such a way that (i), (ii), and (iii) hold whenever $n = 1, 2, \dots, p$. Since φ is Lipschitzean with constant L, (ii) and (iii) apply if $n = p$, and $LM < 1$, for every $y \in Y$ we achieve

$$\begin{aligned} d(f_p(y), \varphi(f_p(y))) &\leq d_H(\varphi(f_{p-1}(y)), \varphi(f_p(y))) \leq Ld(f_{p-1}(y), f_p(y)) \\ &\leq L^p d(f_1(y), f_0(y)) + LM^{1-p} < L^p d(f_1(y), f_0(y)) + M^{-p}, \end{aligned}$$

so that

$$\varphi(f_p(y)) \cap B(f_p(y), L^p d(f_1(y), f_0(y)) + M^{-p}) \neq \emptyset.$$

Because of the assumption $\varphi \in SP(X)$, this produces a continuous function $f_{p+1} : Y \to X$ with the properties:

$$f_{p+1}|_{Y^*} = f^*; \qquad f_{p+1}(y) \in \varphi(f_p(y)) \quad \text{for every } y \in Y;$$
$$d(f_{p+1}(y), f_p(y)) \leq L^p d(f_1(y), f_0(y)) + M^{-p} \quad \text{for all } y \in Y.$$

Thus the existence of the sequence $\{f_n\}$ is established.

We next define, for any $a > 0$, $Y_a = \{y \in Y : d(f_1(y), f_0(y)) < a\}$. Obviously, the family of sets $\{Y_a : a > 0\}$ is an open covering of Y. Moreover, due to (iii) and the completeness of X, the sequence $\{f_n\}$ converges uniformly on each Y_a. Let $f : Y \to X$ be the pointwise limit of $\{f_n\}$. It is easily seen that the function f is continuous. Further, owing to (i) one has $f|_{Y^*} = f^*$. Finally, the range of f is a subset of $Fix(\varphi)$ since, by (ii), $f(y) \in \varphi(f(y))$ for all $y \in Y$. This completes the proof. □

The same arguments used to get Theorem 2.3 actually produce the following more general result.

(2.4) **Theorem.** *Let $\mathcal{D} \subseteq \mathcal{M}$, let X be a complete absolute retract, and let φ be a contractive multivalued map from X into itself, having the selection property with respect to $\mathcal{D}$. Then, for any $Y \in \mathcal{D}$ and any nonempty closed set $Y_0 \subseteq Y$, every continuous function $f_0 : Y_0 \to Fix(\varphi)$ admits a continuous extension over Y.*

Theorem 2.3 has a variety of special cases that are particularly interesting. As an example, Remark 2.2.3 combined with Theorem 2.3 leads to

(2.5) **Theorem.** *Let X be a complete absolute retract and let $\varphi : X \to X$ be a contractive multivalued map. If there exists a Michael family $M(X)$ of subsets of X such that $\varphi(x) \in M(X)$ for all $x \in X$, then the set $Fix(\varphi)$ is an absolute retract.*

Example 1.6.1 and Theorem 2.5 yield the next result (see [15, Theorem 1]).

(2.6) **Theorem.** *Let X be a nonempty, convex, closed subset of a Banach space and let $\varphi : X \to X$ be a contractive multivalued map. Suppose $\varphi(x)$ is convex for all $x \in X$. Then the set $Fix(\varphi)$ is an absolute retract.*

Finally, from Remark 2.2.4, Theorem 2.4, and Proposition 1.1 we infer (see [5, Theorem 1])

(2.7) **Theorem.** *Let X be a retract of $L^1(T, E)$ and let φ be a contractive multivalued map from X into itself, with decomposable values. Then the set $Fix(\varphi)$ is an absolute retract.*

3. Appendix

We now establish the following result, previously announced without proof in Remark 2.2.4.

(3.1) **Theorem.** *Let X be a nonempty closed subset of $L^1(T, E)$ and let $\varphi : X \to X$ be a lower semicontinuous multivalued map, with decomposable values. Then φ has the selection property with respect to the family $\mathcal{D}$ of all separable metric spaces.*

Proof. Throughout this proof, we write θ to denote the zero vector of $L^1(T, E)$ and $\|\cdot\|_1$ in place of $\|\cdot\|_{L^1(T,E)}$. Pick $Y \in \mathcal{D}$ and a pair of continuous functions $f : Y \to X, h : Y \to]0, +\infty[$ such that $\psi(y) = \mathrm{cl}(\varphi(f(y)) \cap B(f(y), h(y))) \neq \emptyset$ for all $y \in Y$. If Y_0 is a nonempty closed subset of Y and g_0 denotes a continuous selection from $\psi|_{Y_0}$, then the function $k_0 : Y_0 \to L^1(T, E)$ defined by

$$k_0(y) = h(y)^{-1}[g_0(y) - f(y)], \qquad y \in Y_0,$$

is a continuous selection of $\eta|_{Y_0}$, where

$$\eta(y) = \mathrm{cl}(h(y)^{-1}[\varphi(f(y)) - f(y)] \cap B(\theta, 1)), \qquad y \in Y.$$

Evidently, the proof is performed as soon as we show that k_0 admits a continuous extension k over Y, with the property $k(y) \in \eta(y)$ for every $y \in Y$.

We first define

$$\xi(y) = \begin{cases} \{k_0(y)\} & \text{if } y \in Y_0, \\ h(y)^{-1}[\varphi(f(y)) - f(y)] & \text{if } y \in Y \setminus Y_0. \end{cases}$$

It is a simple matter to see that the multivalued map $\xi : Y \to L^1(T, E)$ is lower semicontinuous and with decomposable values. Hence, due to Theorem 3 of [6], for any $y \in Y$ and any $u \in \xi(y) \cap B(\theta, 1)$, there exists a continuous selection $k_{y,u} : Y \to L^1(T, E)$ from ξ such that $k_{y,u}(y) = u$. Let

$$V_{y,u} = \{z \in Y : \|k_{y,u}(z)\|_1 < 2^{-1}(1 + \|u\|_1)\}.$$

The family of sets $\{V_{y,u} : y \in Y, u \in \xi(y) \cap B(\theta, 1)\}$ is an open covering of the separable metric space Y, so it has a countable nbd-finite refinement $\{V_n : n \in \mathbb{N}\}$. For each $n \in \mathbb{N}$, choose $y_n \in Y$ and $u_n \in \xi(y_n) \cap B(\theta, 1)$ such that $V_n \subseteq V_{y_n,u_n}$, and define $k_n = k_{y_n,u_n}$. Let $\{p_n\}$ be a continuous partition of unity subordinated to the covering $\{V_n\}$ and let $\{h_n\}$ be a sequence of continuous functions from Y into $[0, 1]$, fulfilling the conditions $h_n(y) = 1$ on $\operatorname{supp} p_n, \operatorname{supp} h_n \subseteq V_n$, $n \in \mathbb{N}$. We now set, for any $y \in Y$,

$$\varphi_n(y)(t) = \|k_n(y)(t)\|, \qquad t \in [0, a] \text{ and } n \in \mathbb{N},$$

$$l(y) = \frac{1}{2}\left[1 - \sum_{n=0}^{\infty} \frac{1 + \|u_n\|_1}{2} p_n(y)\right]^{-1} \sum_{n=1}^{\infty} h_n(y).$$

Since $u_n \in B(\theta, 1)$ and the above summations are locally finite, the function l is well defined, positive, and continuous. Therefore, by Lemma 2 of [6] applied to the sequences $\{\varphi_n\}$ and $\{h_n\}$, and to the function l, there exist a continuous function $\tau : Y \to]0, +\infty[$ and a family $\{A_{\tau,\lambda} : \tau > 0, \lambda \in [0, 1]\}$ of measurable subsets of T satisfying

(a) $A_{\tau,\lambda_1} \subseteq A_{\tau,\lambda_2}$ if $\lambda_1 \le \lambda_2$,
(b) $\mu(A_{\tau_1,\lambda_1} \Delta A_{\tau_2,\lambda_2}) \le |\lambda_1 - \lambda_2| + 2|\tau_1 - \tau_2|$ and $\mu(A_{\tau,\lambda}) = \lambda\mu(T)$,
(c) for each $y \in Y, \lambda \in [0, 1]$, and $n \in \mathbb{N}$, if $h_n(y) = 1$, then

$$\left| \int_{A_{\tau(y),\lambda}} \varphi_n(y)(t)\, d\mu - \lambda \int_T \varphi_n(y)(t)\, d\mu \right| < \frac{1}{4l(y)}.$$

Finally, let us define, for $y \in Y$ and $n \in \mathbb{N}$, $\lambda_0(y) = 0, \lambda_n(y) = \sum_{m \le n} p_m(y)$, $\chi_{y,n} = \chi_{A_{\tau(y),\lambda_n(y)} \setminus A_{\tau(y),\lambda_{n-1}(y)}}$,

$$k(y) = \sum_{n=1}^{\infty} \chi_{y,n} \cdot k_n(y).$$

Bearing in mind condition (b), it is a simple matter to see that the function $k : Y \to L^1(T, E)$ is continuous. Furthermore, for any $y \in Y$ one has $k(y) \in \xi(y)$, because $\xi(y)$ is decomposable. Thus, to complete the proof, we only need to show that $\|k(y)\|_1 < 1$ at all points of Y. Fix $y \in Y$ and observe that, if $I(y) = \{n \in \mathbb{N} : p_n(y) > 0\}$, then $1 \leq \sharp I(y) \leq \sum_{n=1}^{\infty} h_n(y)$. From (a)–(c) we deduce

$$\begin{aligned}
\int_T \|k(y)(t)\| \, d\mu &\leq \sum_{n \in I(y)} \int_{A_{\tau(y),\lambda_n(y)} \setminus A_{\tau(y),\lambda_{n-1}(y)}} \varphi_n(y)(t) \, d\mu \\
&= \sum_{n \in I(y)} \left[\int_{A_{\tau(y),\lambda_n(y)}} \varphi_n(y)(t) \, d\mu - \lambda_n(y) \int_T \varphi_n(y)(t) \, d\mu \right. \\
&\qquad - \int_{A_{\tau(y),\lambda_{n-1}(y)}} \varphi_n(y)(t) \, d\mu \\
&\qquad \left. + \lambda_{n-1}(y) \int_T \varphi_n(y)(t) \, d\mu + p_n(y) \int_T \varphi_n(y)(t) \, d\mu \right] \\
&< \frac{\sharp I(y)}{2l(y)} + \sum_{n=1}^{\infty} \frac{1 + \|u_n\|_1}{2} p_n(y) \leq \frac{1}{2l(y)} \sum_{n=1}^{\infty} h_n(y) + \sum_{n=1}^{\infty} \frac{1 + \|u_n\|_1}{2} p_n(y).
\end{aligned}$$

Hence, owing to the definition of l, $\|k(y)\|_1 < 1$ as required. □

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Added in proof

After this paper was submitted, the authors realized that Theorem 2.3 can be established without the boundedness assumption on the values of φ. Thus, the problem mentioned in the Introduction is now completely solved. The related result is contained in a note to appear in Rend. Circ. Mat. Palermo (2) Suppl. (Proceedings of FAMA '95 Meeting).

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