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FINITE CW COMPLEXES WITH MAXIMAL TORSION GAPS

HUALE HUANG

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Abstract. We investigate some properties of finite CW complexes with maximal homotopy torsion gaps and prove a revision of the Halperin conjecture under an additional condition.

1. Introduction

Let X be a simply connected finite CW complex and $n_X = \max\{i | H_i(X, Q) \neq 0\}$. On the other hand, $\pi_i(X)$ is the direct sum of finitely many copies of $\mathbb{Z}$ and a finite abelian group. We call an interval $[k, l]$ a torsion gap for X if $\pi_k(X)$ and $\pi_l(X)$ both contain copies of $\mathbb{Z}$ and $\pi_i(X)$ $(k < i < l)$ is finite. In [4], S. Halperin proved

Theorem 1. *Let X be a simply connected finite CW complex. If $[k, l]$ is a torsion gap for X, then $l - k < n_X$.*

Then, Halperin proposed the

Conjecture. *If a simply connected finite CW complex X has at least one torsion gap of the form $[k, k + n_X - 1]$, then X has the rational homotopy type of a wedge of n_X-spheres.*

In [6], the author resolved the conjecture in the negative. However, those CW complexes with maximal torsion gaps are very strict. In fact, we have

Theorem A. *If a simply connected finite CW complex X has at least one torsion gap of the form $[k, k + n_X - 1]$, then the k-connected covering $X(k)$ of X has the rational homotopy type of a wedge of spheres.*

Corollary 1. *If X has infinitely many torsion gaps of the form $[k, k + n_X - 1]$, let $k_0 = \min\{k | [k, k + n_X - 1]$ is a torsion gap for $X\}$. Then if $[k, l]$ is a torsion gap for X and $k \geq k_0$, then $l = k + n_X - 1$ and $k \equiv 1 \pmod{n_x - 1}$. In particular, if all the torsion gaps for X have the form $[k, k + n_X - 1]$, then X has the rational homotopy type of a wedge of n_X-spheres.*

Proof. Since $[k_0, k_0 + n_X - 1]$ is a torsion gap for X, the k_0-connected covering $X(k_0)$ of X is $(k_0 + n_X - 2)$-connected. By Theorem 1, there exists $m \leq n_X - 1$, s.t. $\pi_{k_0+n_X-1+m}(X) \supset \mathbb{Z}$. As $k_0 \geq 2$, $(k_0, n_X - 1 + m) + 1 < 2(k_0 + n_X - 1)$; hence

$$H^{k_0+n_X-1}(X(k_0), Q) \neq 0, \qquad H^{k_0+n_X-1+m}(X(k_0), Q) \neq 0.$$

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By Theorem A, $X(k_0)$ has the rational homotopy type of a wedge of spheres. So we have

$$X(k_0)_Q \cong (S^{k_0+n_X-1} \vee S^{k_0+n_X-1+m} \vee Y)_Q.$$

Let $d = (k_0 + n_X - 2, k_0 + n_X - 2 + m)$; then $d|m$. Then there is some integer l s.t. for all $k \geq 0$, $\pi_{l+kd}(X) \supset \mathbb{Z}$. Hence $d = m = n_X - 1$, and $k_0 \equiv 1 \pmod{n_X - 1}$. Using the same argument, we can prove that if $[k, l]$ is a torsion gap for X and $k \geq k_0$, then $l = k + n_X - 1$ and $k \equiv 1 \pmod{n_X - 1}$. □

As a revision of the Halperin conjecture, we have

Theorem B. *If a simply connected finite CW complex X has infinitely many torsion gaps of the form $[k, k + n_X - 1]$, and if $\operatorname{cat}(X) \leq 2$, then X has the rational homotopy type of a wedge of n_X-spheres.*

2. PROOF OF THEOREM A

The main tool in this paper is Sullivan's theory of minimal models. A commutative differential graded algebra (CDGA) over the rational number field Q is a differential graded algebra (A, d), where $A = \sum_{p\geq 0} A^p$, $ab = (-1)^{pq}ba$ for $a \in A^p$, $b \in A^q$, and $d\colon A^p \to A^{p+1}$ is a derivation. Its cohomology is denoted by $H^*(A, d)$. If $V = \sum_{p>0} V^p$ is a graded vector space over Q where V^p is finite dimensional for all p, then we denote by

$$\bigwedge V = \text{Exterior algebra } (V^{\text{odd}}) \otimes \text{ Symmetric algebra } (V^{\text{even}})$$

the free commutative graded algebra on V. A Sullivan minimal model is a CDGA of the form $(\bigwedge V, d)$, where V^i is finite dimensional for $i \geq 1$ and $\operatorname{Im} d \subset (\bigwedge V)^+ \cdot (\bigwedge V)^+$, where $(\bigwedge V)^+$ is the ideal generated by V. To any simply connected space X s.t. each $H_i(X, Q)$ is finite dimensional, $i \geq 2$, Sullivan associates a unique minimal model $(\bigwedge V, d)$, satisfying the following properties:

$$H(\bigwedge V, d) \cong H^*(X, Q); \qquad V \cong \operatorname{Hom}_{\mathbb{Z}}(\pi_*(X), Q).$$

To prove Theorem A, we need two lemmas. Let $(\bigwedge V, d)$ be a minimal model, then the augmentation $\bigwedge V \to Q$ has a minimal model of the form $(\bigwedge V \otimes \bigwedge \overline{V}, d) \cong Q$, in which the differential in the quotient model $Q \otimes \bigwedge_V (\bigwedge V \otimes \bigwedge \overline{V}) = \bigwedge \overline{V}$ is zero and $V^i \cong \overline{V}^{i-1}$. Thus each $\bigwedge V \otimes (\bigwedge \overline{V})^{\leq r}$ is a sub$-(\bigwedge V, d)$ differential model of $\bigwedge V \otimes \bigwedge \overline{V}$. Thus we get a spectral sequence E_i with $E_0^{p,q} = (\bigwedge V)^p \otimes (\bigwedge \overline{V})^q$.

Lemma 1. *If V is finite dimensional and we let $m = \max\{i | V^i \neq 0\}$, then the spectral sequence E_i degenerated at E_{m+1}, that is, $E_{m+1} = E_{m+2} = \cdots = E_\infty$.*

Proof of Lemma 1. It is enough to prove by induction on r that the spectral sequence for $\bigwedge V^{\leq r} \otimes (\bigwedge \overline{V}^{<r})$ collapses at the E_{r+1}-term. The inductive hypothesis then gives the isomorphism

$$(E_{r+1}(\bigwedge V^{\leq r+1} \otimes (\bigwedge \overline{V}^{\leq r})), d_{r+1}) \cong (E_{r+1}(\bigwedge V^{r+1} \otimes (\bigwedge \overline{V}^{r})), d),$$

with $d(\overline{v}) = v$. This clearly implies the result. □

From Lemma 1, we easily get

Corollary 2. *Let V be finite dimensional and $m = \max\{i | V^i \neq 0\}$. If $H^k(\bigwedge V, d) \neq 0$, $H^l(\bigwedge V, d) \neq 0$, and $H^i(\bigwedge V, d) = 0$ for $k < i < l$, then $l - k \leq m$.*

Proof. If $l - k > m$, then by Lemma 1 $E_\infty^{l,0} \neq 0$, which is impossible. $\square$

Like homotopy torsion gap, we define homology torsion gap as follows. Let X be a simply connected CW complex, with finite Betti numbers. If $H_k(X, Q) \neq 0$, $H_l(X, Q) \neq 0$, and $H_i(X, Q) = 0$ for $k < i < l$, then we call $[k, l]$ a homology torsion gap for X.

By Corollary 2, we obtain

Corollary 3. *Let X be a simply connected rational elliptic CW complex and $m = \max\{i | \pi_i(X) \otimes Q \neq 0\}$. If $[k, l]$ is a homology torsion gap for X, then $l - k \leq m$.*

Remark. Corollary 3 is a dual form of Theorem 1 (Halperin), in the sense of Eckman-Hilton duality; furthermore, they are equivalent.

We also need the following lemma, which easily followed from S. Halperin and G. Levin [5].

Lemma 2. *Let X be a simply connected finite CW complex with $(\bigwedge V, d)$ as minimal model. Denote by J the differential ideal of $(\bigwedge V, d)$ defined by $J = (\bigwedge V)^{>n_X} \oplus W$, where W is the complement in $(\bigwedge V)^{n_X}$ of the cocycles. Let $(\bigwedge V^{\leq k}, d)$ be a submodel of $(\bigwedge V, d)$ and $\bigwedge V^{\leq k} \otimes \bigwedge U$ be a minimal of $\bigwedge V^{\leq k} \to Q$. Then the minimal model of $(\bigwedge V^{\leq k} \otimes \bigwedge U) \otimes \bigwedge_{V^{\leq k}} \bigwedge V/J = \bigwedge U \otimes \bigwedge V/J$ has the same minimal model of $X(k)$, the k-connected covering of X.*

A CDGA morphism, ϕ, is a quism if $H(\phi)$ is an isomorphism. The equivalence class of a CDGA (under the equivalence relation generated by quisms) is called its homotopy type. A CDGA is said to be a wedge of spheres if it has the homotopy type of a connected CDGA H with differential zero and satisfies $H^+ \cdot H^+ = 0$.

Now we prove Theorem A.

Proof of Theorem A. For $\bigwedge V^{\leq k} \otimes \bigwedge U$ and $\bigwedge U \otimes \bigwedge V/J$, we have two spectral sequences $\overline{E}_m$, E_m respectively (filtered by the degree of $\bigwedge U$) and morphism $\phi_m : \overline{E}_m \to E_m$ induced by $\bigwedge V^{\leq k} \to \bigwedge V^{\leq k}/J$. Because $\overline{E}_2^{i,*} \to E_2^{i,*}$ is identical for $i < k + n_X$, $\phi_{k+1} : \overline{E}_{k+1}^{i,*} \to E_{k+1}^{i,*}$ is an isomorphism for $i \leq n_X - 1$, while $\overline{E}_{k+1}^{i,*} = 0$ for all $(i, *) \neq (0, 0)$. Thus $E_{k+1}^{i,*} \neq 0$ only for $i = n_X$ or $(i, *) = (0, 0)$. Therefore, we can choose the representatives of $H^+(\bigwedge U \otimes \bigwedge V/J)$ from $\bigwedge U \otimes (\bigwedge V/J)^{n_X}$. Obviously their product is zero; thus $\bigwedge U \otimes \bigwedge V/J$ is a wedge of spheres.

By Lemma 2, we know that the minimal model of $X(k)$ is a wedge of spheres; thus, we get Theorem A. $\square$

3. The second main theorem

Before proving Theorem B, we recall some definitions. In [2], Felix and Halperin define the rational L.S. category, $\mathrm{Cat}_0(X)$, via the Sullivan minimal model $(\bigwedge V, d)$ for X, $\mathrm{Cat}_0(X) \leq m$ if and only if in the diagram

$$\begin{tikzcd}
\bigwedge V \arrow[r, "\pi"] \arrow[dr, "i"'] & \bigwedge V/\bigwedge^{>m+1} V \\
 & \bigwedge V \otimes \bigwedge W \arrow[u, "\phi", "\cong"']
\end{tikzcd}$$

there exists r, so that ri is identity, where π is the projection and ϕ is a quism. When X is a simply connected space with finite Betti numbers, they prove $\mathrm{Cat}_0(X) = \mathrm{Cat}(X_Q)$.

Let L be a graded Lie algebra. Then the radical of L, $\operatorname{rad} L$, is the sum of all resoluble ideals of L.

$$\text{gl. dim}\, L = \text{gl. dim}\, UL = \sup\{i \mid \operatorname{Ext}^i_{UL}(Q, Q) \neq 0\},$$

$$\operatorname{depth} L = \operatorname{depth} UL = \inf\{i \mid \operatorname{Ext}^i_{UL}(Q, UL) \neq 0\}.$$

We need the following theorems from Felix et al. [4] and Felix [1].

Theorem 2. *Let X be a simply connected space with finite Betti numbers. Then*

(1) $\operatorname{depth} H_*(\Omega X, Q) \leq \operatorname{Cat}_0(X) \leq \text{gl. dim}\, H_*(\Omega X, Q)$,
(2) *if* $\operatorname{depth} H_*(\Omega X, Q) = \operatorname{Cat}_0(X)$, *then*

$$\operatorname{Cat}_0(X) = \text{gl. dim}\, H_*(\Omega X, Q).$$

Theorem 3. *Let L be a graded Lie algebra and* $\operatorname{depth} L = m < \infty$. *Then*

(1) $\dim(\operatorname{rad} L)_{\text{even}} \leq m$,
(2) $\operatorname{rad} L$ *is finite dimensional,*
(3) *if* $\dim(\operatorname{rad} L)_{\text{even}} = m$, *then* $L = \operatorname{rad} L$.

Theorem 4. *If* $\operatorname{Cat}_0(X) \leq 2$ *and* $\text{gl. dim}\, H_*(\Omega X, Q) = 2$, *then X is a coformal space, that is, if $(\bigwedge V, d)$ is the Sullivan minimal model, then $d = d_2$.*

Proof of Theorem B. First, $L = \pi_*(\Omega X) \otimes Q$ is a graded Lie algebra by the Samelson Product. We claim that $\operatorname{depth} L \leq 1$. Otherwise depth $= 2 = \operatorname{Cat}_0(X)$; hence, by Theorem 2, $\text{gl. dim}\, L = 2$. According to Theorem 4, X is a coformal space; hence, the Sullivan minimal model has the form $(\bigwedge V, d), d = d_2$. Now let $k_0 = \min\{k \mid [k - n_X + 1, k]$ is a torsion gap for $X\}$ and take $\alpha \in V$ s.t. $|\alpha| > 2k_0$. Then

$$d\alpha = \beta_1\gamma_1 + \beta_2\gamma_2 + \cdots + \beta_m\gamma_m.$$

We may suppose $|\beta_1| = \max\{|\beta_i|, |\gamma_i|\}$. Then $|\beta_1| \geq \frac{|\alpha|}{2} > k_0$; thus $|\beta_1| \equiv 1 \pmod{n_X - 1}$ (by Corollary 1). We can suppose that $d\alpha = \beta_1\gamma + \cdots$ (the other terms having no β_1) in view of $dd\alpha = 0$, $d\gamma \cdot \beta_1 = 0$, hence $d\gamma = 0$ and $|\gamma| = n_X$ because $|\beta_1| \equiv |\alpha| \equiv 1 \mod n_X - 1$. Therefore $X_Q \cong (S^{n_X} \vee Y)_Q$, but this is impossible (because of $\operatorname{depth} \pi_*(\Omega(S^{n_X} \vee Y)) \otimes Q) = 1$). So we have $\operatorname{depth} L \leq 1$.

Next, we claim that if $x \in \pi_*(X) \otimes Q \cong \pi_{*-1}(\Omega X) \otimes Q$ and $|x| \not\equiv 1 \pmod{n_X - 1}$, then $x \in \operatorname{rad} L$.

In fact, we denote by I the Lie ideal generated by x; then we say $I_{\geq k_0} = 0$. Otherwise there is some $\alpha \neq 0$, $\alpha \in I_{\geq k_0}$. Since $I_{\geq k_0}$ is a free Lie algebra, there is some $\omega \in I_{\geq k_0}$ s.t. $[\alpha, \omega] \neq 0$. We write $\alpha = \sum_i [\beta_{1i}, \beta_{2i}]$, with $\beta_{1i} \in I$. The Jacobi identity implies that there is some i and some $u \in I_{\geq k_0}$, s.t. $[\beta_{1i}, u] \neq 0$. Clearly $|\beta_{1i}| < |\alpha|$. We continue the same procedure with β_{1i} in place of α. After finitely many steps, we get $[x, u] \neq 0$, where $|u| \equiv 1 \pmod{n_X - 1}$ and $|u| > k_0$, but $\pi_{|u|+|X|}(X) \otimes Q = 0$. This is a contradiction, so $I_{\geq k_0} = 0$, hence $x \in \operatorname{rad} L$.

We know $\operatorname{depth} L = 1$, by Theorem 3, and $(\operatorname{rad} L)_{\text{even}} = 0$, hence $\pi_{2k+1}(X) \otimes Q = 0$ if $2k+1 \not\equiv 1 \pmod{n_X - 1}$. If n_X is odd, then we must have some $\alpha \in \pi_{n_X}(X) \otimes Q$, and $d\alpha = 0$, so $X_Q = (S^{n_X} \vee Y)_Q$. If $Y_Q \neq \bigvee S^{n_X}$, by the following Lemma 3, X has no torsion gap of the form $[k, k + n_X - 1]$, so that $Y_Q \cong \bigvee S^{n_X}$, and X has the rational homotopy type of a wedge of n_X-spheres. If n_X is even and if $m < n_X$, then $\bigwedge V^{\leq n_X}$ is a polynomial algebra and $dV^{\leq n_X} = 0$. Let $\alpha \in \pi_m(X) \otimes Q$, where $m = \min\{k \mid \pi_k(x) \otimes Q \neq 0\}$, and $\beta \in (\bigwedge V)^{n_X}$ representing a cohomology of $H^{n_X}(\bigwedge V, d)$. Then $d(\alpha\beta) = 0$, $|\alpha\beta| = m + n_X < 2n_X - 1$, so that $\alpha\beta$ cannot be a coboundary; hence $H^{m+n_X}(\bigwedge V, d) \neq 0$. This is in contradiction with the definition of n_X. So $m = n_X$, hence, by the rational Hurewicz Theorem, X has the rational homotopy type of a wedge of n_X-spheres. $\square$

Lemma 3. *If* $X = S^{n_X} \vee Y$ *and* $Y_Q \neq (\bigvee S^{n_X})_Q$, *then* X *has no torsion gap of the form* $[k, k + n_X - 1]$.

Proof. As a Lie algebra, $\pi_*(\Omega X) \otimes Q$ is the free product of $\pi_*(S^{n_X}) \otimes Q$ and $\pi_*(Y) \otimes Q$. It is easy to show that X has no torsion gap of the form $[k, k+n_X-1]$. $\square$

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Institute of Mathematics, Academia Sinica, Beijing 100080, People's Republic of China

Current address: Ph.D. Program in Mathematics, Graduate Center, CUNY, 33 W 42 St., New York, New York 10036

E-mail address: `hhuang@email.gc.cuny.edu`