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THE LOCAL COHOMOLOGY MODULES OF MATLIS REFLEXIVE MODULES ARE ALMOST COFINITE

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Abstract. We show that if M and N are Matlis reflexive modules over a complete Gorenstein local domain R and I is an ideal of R such that the dimension of R/I is one, then the modules $\mathrm{Ext}^i_R(N, \mathrm{H}^j_I(M))$ are Matlis reflexive for all i and j if $\mathrm{Supp}(N) \subseteq V(I)$. It follows that the Bass numbers of $\mathrm{H}^j_I(M)$ are finite. If R is not a domain, then the same results hold for $M = R$.

1. Introduction

Let R be a Noetherian local ring with maximal ideal $\mathfrak{m}$ and residue field k. If M is a finitely generated module, then the local cohomology modules $\mathrm{H}^j_{\mathfrak{m}}(M)$ are artinian, so $\mathrm{Ext}^i_R(k, \mathrm{H}^j_{\mathfrak{m}}(M))$ is finitely generated for every i and j. Grothendieck [G] conjectured the following:

For any ideal I and any finitely generated module M, the module $\mathrm{Hom}_R(R/I, \mathrm{H}^j_I(M))$ is finitely generated for all j.

Here, $\mathrm{H}^j_I(M)$ denotes the j^{th} local cohomology module of M with support in I. Hartshorne [H2] showed that this is false in general. However, he defined an R module M to be *I-cofinite* if $\mathrm{Supp}(M) \subseteq V(I)$ and $\mathrm{Ext}^i_R(R/I, M)$ is finitely generated for all i and asked:

For which rings R and ideals I are the modules $\mathrm{H}^j_I(M)$ I-cofinite for all j, and all finitely generated modules M?

Hartshorne answered this question for the following two cases:

(1) If R is a complete regular local ring and I is a nonzero principal ideal, and
(2) If R is a complete regular local ring and I is a prime ideal with $\dim R/I = 1$.

Hartshorne's work provides motivation for some more recent results on this question. Huneke and Koh extended Hartshorne's second case. Theorem 4.1 of [H-K] shows that if R is a complete Gorenstein domain, I is an ideal of R with $\dim R/I = 1$, and N is finitely generated with support in $V(I)$, then $\mathrm{Ext}^i_R(N, \mathrm{H}^j_I(M))$ is finitely generated for all i and j and for all finitely generated modules M.

This result was further extended in [D], Theorem 3. Delfino was able to remove the Gorenstein condition on R as long as R satisfied one of the three conditions (here K denotes a coefficient ring of R, q the uniformizing parameter of K):

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(i) K is a field;
(ii) K is not a field and $q \in I$;
(iii) K is not a field and q is not an element of any minimal prime of R/I.

Delfino and Marley have eliminated the complete domain hypothesis entirely. They prove in [D-M] that if M is a finitely generated module over a commutative Noetherian ring R and I is an ideal such that $\dim R/I = 1$, then the local cohomology modules $H_I^j(M)$ are I-cofinite for all j.

Hartshorne's work also provided motivation for study in the special case when $M = R$. Theorem 2.3 of [H-K] shows that if R is a regular local ring of characteristic $p > 0$, I is any ideal of R, $j > \operatorname{bight}(I)$, and $\operatorname{Hom}(R/I, \mathrm{H}_I^j(R))$ is finitely generated, then $\mathrm{H}_I^j(R) = 0$. Here, $\operatorname{bight}(I) = \max\{\operatorname{ht}\mathfrak{p} \mid \mathfrak{p} \text{ is a minimal prime ideal of } I\}$.

Huneke and Sharp obtained more information on the structure of $\mathrm{H}_I^j(R)$ under the same conditions on R and I. Theorem 2.1 of [H-S] shows that the Bass numbers of $\mathrm{H}_I^j(R)$ are finite for all j. As a corollary, they show that $\operatorname{Ass}(\mathrm{H}_I^j(R))$ is finite for all j. Using the theory of D-modules, Lyubeznik [L] obtained analagous results for equicharacteristic rings.

The goal of the present paper is to obtain similar results as above, but for a larger class of modules. We first focus our attention to the case when R is a complete Gorenstein domain and I is an ideal of R such that $\dim R/I = 1$. Let $E = E_R(k)$ be the injective hull of the residue field and let $(-)^\vee$ denote the functor $\operatorname{Hom}(-, E)$. Recall that a module is *Matlis reflexive* if $M \cong (M^\vee)^\vee$. The class of Matlis reflexive modules over a complete ring includes all finitely generated and artinian modules. We show that if M and N are Matlis reflexive with $\operatorname{Supp}(N) \subseteq V(I)$, then $\operatorname{Ext}_R^i(N, \mathrm{H}_I^j(M))$ is Matlis reflexive for all i and j. In fact, $\operatorname{Ext}_R^i(N, \mathrm{H}_I^j(M))$ is finitely generated for all i when $j > 0$. We give an example to show that $\operatorname{Ext}_R^i(N, \mathrm{H}_I^0(M))$ need not be finitely generated. However, it turns out that this is enough to show that all the Bass numbers of $\mathrm{H}_I^j(M)$ are finite.

We then remove the domain assumption on R and study the case when $M = R$ and N is finitely generated. In this situation we show that $\operatorname{Ext}_R^i(N, \mathrm{H}_I^j(R))$ is Matlis reflexive. As a corollary, we show that $\mathrm{H}_I^j(R)$ has finite Bass numbers for every j.

2. The domain case

Throughout the rest of the paper, R will denote a Noetherian local ring with maximal ideal $\mathfrak{m}$ and residue field k, I will denote an ideal of R such that $\dim R/I = 1$, E will denote the injective hull of the residue field, and $(-)^\vee$ will denote the functor $\operatorname{Hom}_R(-, E)$. For basic facts about Matlis reflexive modules, we refer the reader to section 3.2 of [St]. In particular, we make extensive use of the following fact:

If R is complete, then an R-module M is Matlis reflexive if and only if there is a short exact sequence

$$0 \to S \to M \to A \to 0$$

with S finitely generated and A artinian (see, for example, [E], Proposition 1.3, or [St], Theorem 3.4.13).

Lemma 1. *If M and N are Matlis reflexive, then $\operatorname{Ext}_R^i(N, \mathrm{H}_I^0(M))$ is Matlis reflexive for all i.*

Proof. Since $\mathrm{H}^0_I(M) \subseteq M$, this follows from Theorem 3 of [Be]. ☐

Theorem 2. *If R is a complete Gorenstein domain, M is Matlis reflexive, and N is finitely generated with $\mathrm{Supp}(N) \subseteq V(I)$, then $\mathrm{Ext}^i_R(N, \mathrm{H}^j_I(M))$ is Matlis reflexive. In fact, $\mathrm{Ext}^i_R(N, \mathrm{H}^j_I(M))$ is finitely generated for all i when $j \geq 1$.*

Proof. Since M is Matlis reflexive, there is a short exact sequence

$$0 \to S \to M \to A \to 0$$

with S finitely generated and A artinian. Since A is artinian, we obtain from the induced long exact sequence of local cohomology modules the exact sequence

$$0 \to \mathrm{H}^0_I(S) \to \mathrm{H}^0_I(M) \to A \to \mathrm{H}^1_I(S) \to \mathrm{H}^1_I(M) \to 0$$

and

$$\mathrm{H}^j_I(M) \cong \mathrm{H}^j_I(S)$$

for $j \geq 2$.

Let K denote the kernel of the map $\mathrm{H}^1_I(S) \to \mathrm{H}^1_I(M)$. Then K is artinian, since A is, so K is Matlis reflexive. Thus, from the short exact sequence

$$0 \to K \to \mathrm{H}^1_I(S) \to \mathrm{H}^1_I(M) \to 0$$

we obtain the long exact sequence

$$\cdots \to \mathrm{Ext}^i_R(N, \mathrm{H}^1_I(S)) \to \mathrm{Ext}^i_R(N, \mathrm{H}^1_I(M)) \to \mathrm{Ext}^{i+1}_R(N, K) \to \cdots .$$

By Theorem 4.1 of [H-K], $\mathrm{Ext}^i_R(N, \mathrm{H}^1_I(S))$ is finitely generated, hence $\mathrm{Ext}^i_R(N, \mathrm{H}^1_I(M))$ is finitely generated if and only if $\mathrm{Ext}^{i+1}_R(N, K)$ is finitely generated.

Since K is artinian, $\mathrm{Hom}(N, K)$ is also artinian, and since $\mathrm{Hom}(N, K) \subseteq \mathrm{Hom}(N, \mathrm{H}^1_I(S))$, $\mathrm{Hom}(N, K)$ has finite length. So $N \otimes K^\vee \cong \mathrm{Hom}(N, K)^\vee$ has finite length and hence $\mathrm{Tor}_i(N, K^\vee)$ has finite length. Thus $\mathrm{Ext}^i(N, K) \cong \mathrm{Tor}_i(N, K^\vee)^\vee$ has finite length for every i. Therefore $\mathrm{Ext}^i_R(N, \mathrm{H}^1_I(M))$ is finitely generated, and hence Matlis reflexive, for all i.

By Theorem 1, $\mathrm{Ext}^i_R(N, \mathrm{H}^0_I(M))$ is Matlis reflexive. For $j \geq 2$,

$$\mathrm{Ext}^i_R(N, \mathrm{H}^j_I(M)) \cong \mathrm{Ext}^i_R(N, \mathrm{H}^j_I(S))$$

for all i, and so by Theorem 4.1 of [H-K], $\mathrm{Ext}^i_R(N, \mathrm{H}^j_I(M))$ is finitely generated, hence Matlis reflexive. ☐

We note that $\mathrm{Ext}^i_R(N, \mathrm{H}^0_I(M))$ need not be finitely generated. For example, let $N = R/I$, $M = E$, and $i = 0$. Then $\mathrm{Ext}^i_R(N, \mathrm{H}^0_I(M)) = (R/I)^\vee$. Since $\dim R/I = 1$, $(R/I)^\vee$ is not finitely generated. However, the following corollary shows that this is not too bad.

Recall that the i^{th} term $E^i(M)$ in the minimal injective resolution of an R-module M is uniquely determined, up to isomorphism, by M. In fact, there is a family $\mathcal{F}$ of prime ideals of R such that $E^i(M) \cong \bigoplus_{\mathfrak{q} \in \mathcal{F}} E(R/\mathfrak{q})$. The i^{th} *Bass number with respect to* $\mathfrak{p}$, denoted $\mu_i(\mathfrak{p}, M)$, of M is defined to be the number of copies of the injective hull of $R/\mathfrak{p}$ in this direct sum decomposition of $E^i(M)$. By 2.7 of [B], this is the same as the vector space dimension of $(\mathrm{Ext}^i_R(R/\mathfrak{p}, M))_{\mathfrak{p}}$ over $(R/\mathfrak{p})_{\mathfrak{p}}$. We refer to the numbers $\mu_i(\mathfrak{p}, M)$ for all i and all $\mathfrak{p}$ collectively as the Bass numbers of M.

Corollary 3. *If R is a complete Gorenstein domain and M is a Matlis reflexive R-module, then the Bass numbers of $\mathrm{H}_I^j(M)$ are finite for every j.*

Proof. Let k be the residue field of R. Then $\mathrm{Ext}_R^i(k, \mathrm{H}_I^j(M))$ is Matlis reflexive by Theorem 2. Since $\mathrm{Ext}_R^i(k, \mathrm{H}_I^j(M))$ is also a k vector space, it must be finitely generated. If $\mathfrak{p}$ is any nonmaximal prime, it follows from Proposition 1.3 of [E] that $M_\mathfrak{p}$ is finitely generated over $R_\mathfrak{p}$. Since $\dim R/I = 1$, we have $(\mathrm{H}_I^j(M))_\mathfrak{p} \cong \mathrm{H}_{\mathfrak{p}R_\mathfrak{p}}^j(M_\mathfrak{p})$ if $\mathfrak{p} \supseteq I$ or $(\mathrm{H}_I^j(M))_\mathfrak{p} \cong 0$ if $\mathfrak{p} \not\supseteq I$. In either case, it follows that $(\mathrm{Ext}_R^i(R/\mathfrak{p}, \mathrm{H}_I^j(M)))_\mathfrak{p}$ is finitely generated over $R_\mathfrak{p}$. □

Lemma 4. *Let R be a complete Gorenstein domain, M a Matlis reflexive module, and N an artinian module. Then $\mathrm{Ext}_R^i(N, \mathrm{H}_I^j(M))$ is finitely generated (and thus Matlis reflexive) for all i and j.*

Proof. Since M is Matlis reflexive, $\mathrm{H}_I^j(M)$ has finite Bass numbers by Corollary 3. Fix j. Let $0 \to \mathrm{H}_I^j(M) \to \mathbf{J}^*$ be a minimal injective resolution of $\mathrm{H}_I^j(M)$. Then for each t,

$$\mathrm{Hom}(N, J^t) \cong \bigoplus \mathrm{Hom}(N, E)$$

since N is artinian. Since $\mathrm{H}_I^j(M)$ has finite Bass numbers, the direct sum is a finite direct sum. By Matlis duality, $\mathrm{Hom}(N, E)$ is finitely generated. Thus $\mathrm{Hom}(N, J^t)$, and hence $\mathrm{Ext}_R^i(N, \mathrm{H}_I^j(M))$, is finitely generated. □

Theorem 5. *If R is a complete Gorenstein domain and M and N are Matlis reflexive such that $\mathrm{Supp}(N) \subseteq V(I)$, then $\mathrm{Ext}_R^i(N, \mathrm{H}_I^j(M))$ is Matlis reflexive for all i and j. In fact, $\mathrm{Ext}_R^i(N, \mathrm{H}_I^j(M))$ is finitely generated for all i when $j > 0$.*

Proof. For $j = 0$, this is just Lemma 1. Fix $j > 0$. Since N is Matlis reflexive, there is a short exact sequence

$$0 \to T \to N \to B \to 0$$

with T finitely generated and B artinian. This induces a long exact sequence

$$\cdots \to \mathrm{Ext}_R^i(B, \mathrm{H}_I^j(M)) \to \mathrm{Ext}_R^i(N, \mathrm{H}_I^j(M)) \to \mathrm{Ext}_R^i(T, \mathrm{H}_I^j(M)) \to \cdots .$$

By Theorem 2, $\mathrm{Ext}_R^i(T, \mathrm{H}_I^j(M))$ is finitely generated. By Lemma 4, $\mathrm{Ext}_R^i(B, \mathrm{H}_I^j(M))$ is finitely generated. Thus $\mathrm{Ext}_R^i(N, \mathrm{H}_I^j(M))$ is finitely generated, and hence Matlis reflexive for all i when $j > 0$. □

3. The nondomain case

In this section we drop the domain requirement on the ring R and obtain similar results for the local cohomology modules of R. As one of our main tools, we use Grothendieck's local duality theorem. In the special case when R is a Gorenstein ring with $\dim R = d$, local duality states that, for any finitely generated R-module M,

$$\mathrm{H}_\mathfrak{m}^i(M) \cong \mathrm{Ext}_R^{d-i}(M, R)^\vee$$

for every i. For details the reader is referred to [H1].

Lemma 6. *If R is a complete Gorenstein ring and N is finitely generated with $\mathrm{Supp}(N) \subseteq V(I)$, then $\mathrm{Ext}_R^i(N, \mathrm{H}_I^j(R))$ is Matlis reflexive for all i and j.*

Proof. By Lemma 2 of [D], we may assume that $N = R/I$. Let $0 \to R \to \mathbf{D}^*$ be an injective resolution of R. Since R is Gorenstein, $D^t = (\bigoplus E(R/\mathfrak{p});\ \mathrm{ht}\,\mathfrak{p} = t)$, so $\mathrm{H}^0_I(\mathbf{D}^*)$ is the complex

$$0 \to E' \xrightarrow{\psi} E \to 0$$

where $E' = (\bigoplus E(R/\mathfrak{p});\ \mathrm{ht}\,\mathfrak{p} = d-1,\ \mathfrak{p} \supseteq I)$, E is the injective hull of the residue field, and $d = \dim R$. Thus $\mathrm{H}^j_I(R) = 0$ for $j \neq d-1, d$, and there is a short exact sequence

$$0 \to \mathrm{H}^{d-1}_I(R) \to E' \to A \to 0$$

where A is the image of ψ. It follows that there is an exact sequence

$$\begin{aligned} 0 &\to \mathrm{Hom}(R/I, \mathrm{H}^{d-1}_I(R)) \to \mathrm{Hom}(R/I, E') \\ &\to \mathrm{Hom}(R/I, A) \to \mathrm{Ext}^1(R/I, \mathrm{H}^{d-1}_I(R)) \to 0 \end{aligned}$$

and that $\mathrm{Ext}^i(R/I, \mathrm{H}^{d-1}_I(R)) \cong \mathrm{Ext}^{i-1}(R/I, A)$ for $i > 1$. Since A is artinian, it follows that $\mathrm{Ext}^i(R/I, \mathrm{H}^{d-1}_I(R))$ is artinian, and hence Matlis reflexive, for $i > 1$.

Consider the complex

$$0 \to \mathrm{Hom}(R/I, E') \xrightarrow{\mathrm{Hom}(R/I,\psi)} \mathrm{Hom}(R/I, E) \to 0.$$

Let J denote the image of $\mathrm{Hom}(R/I, \psi)$ and K denote the kernel of $\mathrm{Hom}(R/I, \psi)$. Then $K = \mathrm{Ext}^{d-1}_R(R/I, R)$ and $\mathrm{Hom}(R/I, E)/J = \mathrm{Ext}^d_R(R/I, R)$. Using the left exactness of $\mathrm{Hom}(R/I, -)$ and local duality, we have

$$\mathrm{Hom}(R/I, \mathrm{H}^{d-1}_I(R)) \cong K \cong (\mathrm{H}^1_{\mathfrak{m}}(R/I))^\vee$$

and

$$\mathrm{Ext}^1(R/I, \mathrm{H}^{d-1}_I(R)) \cong \mathrm{Hom}(R/I, A)/J \subseteq \mathrm{Hom}(R/I, E)/J \cong (\mathrm{H}^0_{\mathfrak{m}}(R/I))^\vee.$$

Since R is complete and $\mathrm{H}^s_{\mathfrak{m}}(R/I)$ is artinian for any s, we have $\mathrm{Ext}^i(R/I, \mathrm{H}^{d-1}_I(R))$ is finitely generated for $i = 0, 1$, and hence Matlis reflexive.

Finally, note that $\mathrm{H}^d_I(R)$ is a factor of E, hence artinian. Therefore, $\mathrm{Ext}^i(R/I, \mathrm{H}^d_I(R))$ is artinian, and hence Matlis reflexive, for all i. □

Corollary 7. *If R is a complete Gorenstein ring, then $\mathrm{H}^j_I(R)$ has finite Bass numbers for every j.*

Proof. By Lemma 6, $\mathrm{Ext}^i_R(k, \mathrm{H}^j_I(R))$ is Matlis reflexive. The proof then proceeds as in Corollary 3. □

Lemma 8. *If R is a complete Gorenstein ring and N is an artinian module, then the module $\mathrm{Ext}^i_R(N, \mathrm{H}^j_I(R))$ is finitely generated for all i and j.*

Proof. Since $\mathrm{H}^j_I(R)$ has finite Bass numbers, the proof of Lemma 4 may be adapted. □

Theorem 9. *If R is a complete Gorenstein ring and N is a Matlis reflexive module with $\mathrm{Supp}(N) \subseteq V(I)$, then $\mathrm{Ext}^i_R(N, \mathrm{H}^j_I(R))$ is Matlis reflexive for all i and j.*

Proof. Since N is Matlis reflexive, there is a short exact sequence

$$0 \to T \to N \to B \to 0$$

with T finitely generated and B artinian. This induces a long exact sequence

$$\cdots \to \mathrm{Ext}^i_R(B, \mathrm{H}^j_I(R)) \to \mathrm{Ext}^i_R(N, \mathrm{H}^j_I(R)) \to \mathrm{Ext}^i_R(T, \mathrm{H}^j_I(R)) \to \cdots.$$

By Lemma 6, $\mathrm{Ext}^i_R(T, \mathrm{H}^j_I(R))$ is Matlis reflexive for all j. By Lemma 8, $\mathrm{Ext}^i_R(B, \mathrm{H}^j_I(R))$ is finitely generated for all j. Thus $\mathrm{Ext}^i_R(N, \mathrm{H}^j_I(R))$ is Matlis reflexive for all i and j. □

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