PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 124, Number 9, September 1996

THE TWO-CARDINALS TRANSFER PROPERTY AND RESURRECTION OF SUPERCOMPACTNESS

SHAI BEN-DAVID AND SAHARON SHELAH

(Communicated by Andreas R. Blass)

ABSTRACT. We show that the transfer property $(\aleph_1, \aleph_0) \to (\lambda^+, \lambda)$ for singular λ does not imply (even) the existence of a non-reflecting stationary subset of λ^+. The result assumes the consistency of ZFC with the existence of infinitely many supercompact cardinals. We employ a technique of "resurrection of supercompactness". Our forcing extension destroys the supercompactness of some cardinals; to show that in the extended model they still carry some of their compactness properties (such as reflection of stationary sets), we show that their supercompactness can be resurrected via a tame forcing extension.

1. INTRODUCTION

The results presented in this paper extend our previous work on the relative strength of combinatorial properties of successors of singular cardinals.

In a seminal paper [J72] Jensen has presented a collection of combinatorial properties that hold in the constructible universe **L**. From the point of view of applications of set theory to other branches of mathematics, these properties are "all you have to know about **L**". Ever since that paper, these properties were applied to a wide spectrum of questions to provide consistency results inside set theory as well as in other branches of mathematics ([Sh:44], [E80], [F83], to mention just a few).

It seems natural to ask to what degree can these properties replace the axiom $\mathbf{V} = \mathbf{L}$? Is there any combinatorial principle that implies all these properties? What is the relative strength of these properties? What are the implication relations among them?

The picture seems to be basically settled for limit cardinals and for successors of regular cardinals, [Mi72], [G76]. Our investigations have focused on successors of singular cardinals. Essentially we have been able to prove, assuming the consistency of the existence of large cardinals, that all the nontrivial implications among these properties are not provable in ZFC (see [BdSh:203], [BdSh:236], [BM86]).

Here we examine the strength of the model theoretic two-cardinals transfer property[1] $(\aleph_1, \aleph_0) \to (\lambda^+, \lambda)$. Jensen [J72] has shown that it is implied by $\square_\lambda$. A quite straightforward argument can show that it implies the weaker $\square^*_\lambda$ principle.

Received by the editors December 14, 1989 and, in revised form, March 13, 1995.
1991 *Mathematics Subject Classification.* Primary 03E35, 03E55, 04A20.

[1]Which means: if a first-order sentence ψ (with a unary predicate P) has a model M such that $\|M\| = \aleph_1, |P^M| = \aleph_0$, then it has a model N with $\|N\| = \lambda^+, |P^N| = \lambda$.

We show that $(\aleph_1, \aleph_0) \to (\lambda^+, \lambda)$ for a singular λ does not imply the existence of a non-reflecting stationary subset of λ^+ (as long as ZFC is consistent with the existence of infinitely many supercompact cardinals). It follows that the implication from $\Box_\lambda$ to $(\aleph_1, \aleph_0) \to (\lambda^+, \lambda)$ is strict and that $\Box^*_\lambda$ (or, equivalently, the existence of a special λ^+-Aronszajn tree) does not imply the existence of a non-reflecting stationary subset of λ^+.

We use a technique which we call *resurrection of supercompactness*. (The idea of "resurrecting" a large cardinal property in a further forcing extension probably first occurred in Kunen [K78].) We start with a model $\mathbf{V}$ in which λ is a limit of supercompact cardinals and therefore all stationary subsets of λ^+ reflect. We extend it through forcing to a model $\mathbf{V}[G]$ in which the two-cardinal transfer property holds.

Now we have to argue why we still have reflection of all stationary subsets of λ^+ (although our forcing has inevitably destroyed the supercompactness of a final segment of cardinals below λ). Instead of applying the commonly used combinatorial analysis to our forcing partial order, we demonstrate the reflection property by showing that we could "resurrect" the supercompactness of any cardinal ρ below λ by a further forcing extension $\mathbb{Q}_\rho$ that preserves reflection of appropriate subsets of λ^+.

2. Proof outline

The proof is based on a translation of the transfer property to a combinatorial principle $\mathcal{S}_\lambda$. We show how $\mathcal{S}_\lambda$ (and therefore $(\aleph_1, \aleph_0) \to (\lambda^+, \lambda)$) can be forced using a "mild" forcing notion. The mildness of the forcing notion guarantees that over certain models, where every stationary subset of λ^+ reflects, such a forcing extension would not destroy the reflection.

The natural candidate for exhibiting reflection of all stationary subsets of λ^+ is a model in which λ is a limit of supercompact cardinals. Letting $\mathbf{V}$ be such a model, standard compactness arguments show that $\Box^*_\lambda$ fails, and therefore $(\aleph_1, \aleph_0) \to (\lambda^+, \lambda)$ fails in $\mathbf{V}$. It follows that if we extend $\mathbf{V}$ to a model $\mathbf{V}[G]$ of $(\aleph_1, \aleph_0) \to (\lambda^+, \lambda)$ the supercompactness of a final segment of the cardinals below λ will be destroyed. We wish to show that our extension was mild enough to retain some of the supercompactness consequences—namely, the reflection of all stationary subsets of λ^+.

To this end we use a technique we call *resurrection of supercompactness*. We further extend $\mathbf{V}[G]$ to a model $\mathbf{V}[G][H]$. We show that in $\mathbf{V}[G][H]$ the supercompactness of certain cardinals is resurrected. Consequently, in $\mathbf{V}[G][H]$ we do have the desired reflection principle. All that is left to do is to make sure that reflection of some stationary $S \subseteq \lambda^+$ in $\mathbf{V}[G][H]$ can only occur if S was already a reflecting stationary subset of λ^+ in $\mathbf{V}[G]$.

More precisely, for every supercompact cardinal ρ below λ, we establish the existence of a forcing notion $\mathbb{Q}_\rho$ such that:

(i) $\mathbb{Q}_\rho$ preserves stationarity of subsets of $S^{\lambda^+}_\rho = \{\alpha < \lambda^+ : \mathrm{cf}(\alpha) < \rho\}$,
(ii) the extension by $[G][G_{\mathbb{Q}_\rho}]$ preserves the supercompactness of ρ.

As λ is a singular limit of cardinals, given any stationary subset S of λ^+ (in $\mathbf{V}[G]$), there is some supercompact ρ for which $S \cap S^{\lambda^+}_\rho$ is stationary in λ^+. In $\mathbf{V}[G][G_{\mathbb{Q}_\rho}]$, ρ is a supercompact cardinal and, by property (i), $S \cap S_{\rho\lambda^+}$ is still stationary so it reflects, i.e. for some $\alpha < \lambda^+$, $S \cap S_{\rho\lambda^+} \cap \alpha$ is stationary in α. It follows that S reflects in $\mathbf{V}[G]$.

3. The principle $\mathcal{S}_\lambda$

In [Sh:269] Shelah introduced a principle $\mathcal{S}_\lambda$ related to $\square$. This principle captures in a combinatorial formulation the model theoretic transfer property

$$(\aleph_1, \aleph_0) \to (\lambda^+, \lambda).$$

Definition 1. $\mathcal{S}_\lambda$ asserts the existence of a sequence

$$\langle C^i_\alpha : \alpha < \lambda^+, i < \mathrm{cf}(\lambda)\rangle$$

such that:

(1) For every $\alpha < \lambda^+$

$$\alpha = \bigcup_{i<\mathrm{cf}(\lambda)} C^i_\alpha \text{ and } i < j < \mathrm{cf}(\lambda) \text{ imply } C^i_\alpha \subseteq C^j_\alpha.$$

(2) For every $i < \mathrm{cf}(\lambda)$, $\sup\{|C^i_\alpha| : \alpha < \lambda^+\} < \lambda$.
(3) For $\alpha < \beta$, $\alpha \in C^i_\beta$ implies $C^i_\alpha = C^i_\beta \cap \alpha$.

Theorem 2 (Shelah)**.** *For a strong limit singular λ,*

$$\mathcal{S}_\lambda \textit{ is equivalent to } (\aleph_1, \aleph_0) \to (\lambda^+, \lambda).$$

(*Actually both properties are equivalent to a seemingly stronger transfer property.*) □

We refer the reader to [Sh:269] for the full theorem and its proof. To gain a feeling for the content of the new $\mathcal{S}_\lambda$ principle let us demonstrate its strength by proving the following corollary of the above theorem directly.

Corollary 3. *For a strong limit singular cardinal λ, $\mathcal{S}_\lambda$ implies $\square^*_\lambda$.*

Proof. For a set t of ordinals let $\overline{t}$ be the closure of t in $\sup(t)$, i.e. $\overline{t} = \{\alpha : \alpha \in t$ or $\alpha = \sup(\alpha \cap t) < \sup(t)\}$.

Let $\langle C^i_\alpha : \alpha < \lambda^+, i < \mathrm{cf}(\lambda)\rangle$ be a $\mathcal{S}_\lambda$ sequence. Define A_α to be

$$\bigcup_{i<\mathrm{cf}(\lambda)} \{s : \text{ there is an increasing sequence } t \subseteq C^i_\alpha \text{ such that } |t| < \lambda \text{ and}$$
$$s = \overline{t} \text{ (the closure of } t \text{ in } \alpha)\} \cup \{s \subseteq \alpha : |s| < \mathrm{cf}(\lambda)\}.$$

As each C^i_α has cardinality less than λ and λ is a strong limit cardinal, we have $|A_\alpha| \leq \lambda$ for all α. Let us see that the sequence $\langle A_\alpha : \alpha < \lambda^+\rangle$ is a $\square^*_\lambda$-sequence. For any $\delta < \lambda^+$, if $\mathrm{cf}(\lambda) < \mathrm{cf}(\delta)$, then for some i, C^i_δ is an unbounded subset of δ. Let t_δ be any increasing sequence of members of such a C^i_δ such that $\overline{t}_\delta$ (the closure of t_δ in δ) has order type $\mathrm{cf}(\delta)$ and is unbounded in δ. For any limit point β of t_δ, for some $j \in [i, \mathrm{cf}(\lambda))$, $\beta \in C^j_\delta$ and therefore $t_\delta \cap \beta \subseteq C^j_\beta$ (as $C^j_\beta = C^j_\delta \cap \beta$), so $\overline{t_\delta \cap \beta}$ is a member of A_β. But $\overline{t_\delta \cap \beta} = \overline{t}_\delta \cap \beta$. We still have to handle ordinals δ of cofinality $\leq \mathrm{cf}(\lambda)$, but for such an ordinal we can pick any continuous sequence increasing to δ, say t_δ, such that $\mathrm{otp}(t_\delta) = \mathrm{cf}(\delta) \leq \mathrm{cf}(\lambda)$ and then for any $\beta < \delta$, $t_\delta \cap \beta$ is a subset of β of cardinality less than $\mathrm{cf}(\delta)$, so it is a member of A_β. □

Theorem 4. *Assuming ZFC is consistent with the existence of infinitely many supercompact cardinals, there is a model of ZFC with a singular strong limit cardinal λ for which $\mathcal{S}_\lambda$ holds and every stationary subset of λ^+ reflects.*

Proof. Let λ be a singular limit of supercompact cardinals, so $2^\lambda = \lambda^+$. It follows that λ is a strong limit cardinal and that every stationary subset of λ^+ reflects. We define a forcing notion $\mathbb{P}'$ such that $\mathcal{S}_\lambda$ holds in $\mathbf{V}^{\mathbb{P}'}$. We will show that in $\mathbf{V}^{\mathbb{P}'}$, λ is still a strong limit and every stationary subset of λ^+ reflects. By iterating Laver's indestructibility forcing [L78] we may assume that λ is a limit of an increasing sequence of supercompact cardinals $\langle \lambda_i : i < \mathrm{cf}(\lambda)\rangle$ such that for each $i < \mathrm{cf}(\lambda)$ if $\mathbb{Q}$ is a λ_i-directed-closed forcing notion, then λ_i remains supercompact after forcing with $\mathbb{Q}$. □

Definition of $\mathbb{P}$. *A condition p in* $\mathbb{P}$ is an initial segment of an $\mathcal{S}_\lambda$-sequence, i.e., for some $\beta < \lambda^+$,

$$p = \{C^i_\alpha : \alpha \leq \beta, i < \mathrm{cf}(\lambda)\}$$

where the C^i_α's satisfy the demands (1) and (3) from the definition of an $\mathcal{S}_\lambda$ sequence and demand (2) is replaced by $|C^i_\alpha| \leq \lambda_i$. We call β *the domain* of the condition $p, \beta = \mathrm{dom}(p)$.

For $p, q \in \mathbb{P}$ we say that $p \leq q$ if and only if

$$\mathrm{dom}(p) \subseteq \mathrm{dom}(q) \quad \text{and} \quad p = q \restriction \mathrm{dom}(p)$$

(where $q \restriction \mathrm{dom}(p)$ denotes the restriction of q to $\mathrm{dom}(p)$).

The forcing notion $\mathbb{P}$ is the natural candidate for introducing an $\mathcal{S}_\lambda$-sequence. We do not know how to guarantee reflection of stationary sets in the model obtained by forcing with $\mathbb{P}$. To obtain the model we are aiming for we shall later apply a further forcing extension.

Lemma 5. *For $p \in \mathbb{P}$ and $\gamma = \mathrm{dom}(p) + 1$ there is a condition $q \in \mathbb{P}$, $p \leq q$ such that $\mathrm{dom}(q) = \gamma$.*

Proof. As q is to extend p, its sequence $\langle C^i_\alpha : i < \mathrm{cf}(\lambda), \alpha < \gamma\rangle$ is already determined and we have to define only $\langle C^i_\gamma : i < \mathrm{cf}(\lambda)\rangle$. Let $C^i_\gamma = \{\beta\} \cup C^i_\beta$ where $\beta = \mathrm{dom}(p)$ (so $\gamma = \beta + 1$). As for all $i < \mathrm{cf}(\lambda)$, we get $\beta \cap C^i_\gamma = C^i_\beta$; it is trivial to check that q is a condition in $\mathbb{P}$. □

We would like to have some closure properties for $\mathbb{P}$. The next lemma shows that under some circumstances an increasing chain of conditions in $\mathbb{P}$ is guaranteed to have an upper bound.

Lemma 6. *Let $\langle p_j : j < \delta\rangle$ be an increasing sequence of conditions in $\mathbb{P}$, $\beta_j = \mathrm{dom}(p_j)$ and $\beta = \lim_{j<\delta} \beta_j$. For each $\alpha < \beta$ let $\langle C^i_\alpha : i < \mathrm{cf}(\lambda)\rangle$ be such that whenever $\alpha \in \mathrm{dom}(p_j)$ this is the α's sequence in p_j (as the p_j's form an increasing chain, this is well defined).*

If *there is an unbounded $C \subseteq \beta$, such that for every $\alpha < \gamma$ from C there is $i < \mathrm{cf}(\lambda)$ such that $C^i_\alpha = C^i_\gamma \cap \alpha$ (or for every γ in C, $C \cap C^i_\gamma$ for some i)*, then *there is a condition $q \in \mathbb{P}$ such that $p_j \leq q$ for all $j < \delta$.*

Proof. Let

$$q = \langle C^i_\alpha : \alpha \leq \beta, i < \mathrm{cf}(\lambda)\rangle.$$

As q extends all the p_j's all we have to define is $\langle C^i_\beta : i < \mathrm{cf}(\lambda)\rangle$. We may assume that $\mathrm{otp}(C) = \mathrm{cf}(\beta)$ as otherwise we may replace C with such an unbounded closed subset. Let λ_{i_0} be the first element of $\langle \lambda_i : i < \mathrm{cf}(\lambda)\rangle$ above $\mathrm{cf}(\beta)$. For $i < i_0$ let $C^i_\alpha = \varnothing$, and for $i_0 \le i$ let $C^i_\beta = \bigcup_{\alpha \in C} C^i_\alpha$. As $\bigcup_{i<\mathrm{cf}(\lambda)} C^i_\alpha = \alpha$ for all $\alpha \in C$ and $\langle C^i_\alpha : i < \mathrm{cf}(\lambda)\rangle$ is increasing, we have $\bigcup_{i \in [i_0, \mathrm{cf}(\lambda))} C^i_\alpha = \alpha$ (for $\alpha \in C$). Since $\bigcup_{\alpha \in C} \alpha = \beta$, we get $\bigcup_{i<\mathrm{cf}(\lambda)} C^i_\beta = \beta$. As for each α the sequence $\langle C^i_\alpha : i < \mathrm{cf}(\lambda)\rangle$ is increasing, so is $\langle C^i_\alpha : i < \mathrm{cf}(\lambda)\rangle$. As each λ_i is a regular cardinal, $|C^i_\alpha| \le \lambda_i$ and $\mathrm{otp}(C) \le \lambda_{i_0}$, we get $|C^i_\beta| \le \lambda_i$ for all $i \in [i_0, \mathrm{cf}(\lambda))$ and for $i < i_0$ this is trivial.

We are left with the coherence demand (3). Assume that $\delta \in C^i_\beta$ and let γ be the first member of C above δ and $i \ge i_0$. Since $C^i_\beta = \bigcup_{\alpha \in C} C^i_\alpha$, we have $C^i_\beta \cap \gamma = \bigcup_{\alpha \in C} C^i_\alpha \cap \gamma$. For $\alpha > \gamma$ we have $C^i_\alpha \cap \gamma = C^i_\gamma$, while $C^i_\alpha \subseteq C^i_\gamma$ for $\alpha < \gamma$ (as $C^i_\gamma \cap \alpha = C^i_\alpha$). Thus we may conclude that $C^i_\beta \cap \gamma = C^i_\gamma$ and $\delta \in C^i_\gamma$. Pick some p_ρ in the sequence of conditions such that $\mathrm{dom}(p_\rho) \ge \gamma$, so $p_\rho(\gamma) = \langle C^i_\gamma : i < \mathrm{cf}(\lambda)\rangle$. Since $\delta \in C^i_\gamma$, we know that $p_\rho \Vdash$ "$C^i_\gamma \cap \delta = C^i_\delta$". For all p_j with $j > \rho$ we have $p_j(\delta) = p_\rho(\delta)$ and this is $q(\delta) = \langle C^i_\delta : i < \mathrm{cf}(\lambda)\rangle$. It follows that $q \Vdash$ "$C^i_\beta \cap \delta = C^i_\delta$", as needed. □

Lemma 7. *The forcing notion $\mathbb{P}$ is μ-strategically closed for each $\mu < \lambda$.*

Proof. Given any such μ we have to define a strategy for Player I such that if an increasing sequence of conditions $\langle p_i : i < \delta\rangle$ (where $\delta \le \mu$) is constructed and for any even and limit $\rho < \mu$, p_ρ is defined by applying our strategy to $\langle p_i : i < \rho\rangle$, then there is a condition p_δ above all members of the sequence.

Denote by i_0 the first i such that $\lambda_i > \mu$ (λ_j's are as defined in the proof of Lemma 6). Our strategy will have the property (where i_0 is as above)

(⊛) If $\rho_1 < \rho_2 < \mu$ and p_{ρ_1}, p_{ρ_2} are both defined by the strategy, then for all $i < \mathrm{cf}(\lambda)$

$$C^i_{\mathrm{dom}(p_{\rho_1})} = C^i_{\mathrm{dom}(p_{\rho_2})} \cap \mathrm{dom}(p_{\rho_1}) \quad \text{and} \quad i \le i_0 \Rightarrow C^i_{\mathrm{dom}(p_{\rho_1})} = \varnothing = C^i_{\mathrm{dom}(p_{\rho_2})}.$$

Let us define the strategy.

Case (i): *ρ is a limit ordinal.* The sequence $E_\rho = \langle \mathrm{dom}(p_i) : i \text{ is even or limit}\rangle$ is increasing and unbounded in $\mathrm{dom}(p_\rho)$ and the condition (⊛) holds along it. We use the proof of Lemma 6 to define p_ρ. Note that the definition of Lemma 6 does satisfy (⊛) for $\rho_1, \rho_2 \in E_\rho \cup \{\rho\}$.

Case (ii): $\rho = 2$. Let p_ρ be any one-level extension of ρ_1. By Lemma 5 such an extension exists.

Case (iii): *ρ is a successor ordinal.* Let γ be $\mathrm{dom}(p_{\rho-1})$ and let ζ^* be the maximal $\zeta < \rho$ for which p_ζ is defined by the strategy (of Player I). Such a ζ^* always exists as Player I gets to play at limit stages. Let $\beta = \mathrm{dom}(p_{\zeta^*})$. As $p_{\rho-1}$ extends p_{ζ^*}, for all $\alpha \le \beta$ we have $p_{\rho-1}(\alpha) = p_{\zeta^*}(\alpha)$ and let us denote it by $\langle C^i_\alpha : i < \mathrm{cf}(\lambda)\rangle$. For $\beta < \alpha \le \gamma$ let $\langle C^i_\alpha : i < \mathrm{cf}(\lambda)\rangle$ be $p_{\rho-1}(\alpha)$.

Now, p_ρ will be a one-level extension of $p_{\rho-1}$, so we have to define only its last level $p_\rho(\gamma+1) = \langle C^i_{\gamma+1} : i < \mathrm{cf}(\lambda)\rangle$. Let j be the first such that $\beta \in C^j_\gamma$. For $i < i_0$ let $C^i_{\gamma+1} = \varnothing$, for $i_0 \le i < j$ let $C^i_{\gamma+1} = C^i_\beta$ and finally for $j \le i$ let $C^i_{\gamma+1} = \{\gamma\} \cup C^i_\gamma$. It is easy to check that the condition (⊛) is satisfied and the p_ρ thus defined is a condition in $\mathbb{P}$ extending $p_{\rho-1}$.

To show that this strategy works we just have to invoke Lemma 6 and, by (⊛), E_ρ contains a set C as assumed by the lemma. □

Lemma 8. *For each* $p \in \mathbb{P}$ *such that* $\operatorname{dom}(p) < \alpha < \lambda^+$ *there is an extension* q *of* p *such that* $\operatorname{dom}(q) \geq \alpha$.

Proof. Assume, by way of contradiction, that there is a $\beta < \lambda^+$ so that there is some p with no extension q of p satisfying $\operatorname{dom}(q) \geq \beta$. Let β_0 be the first such β and p such a condition. If β_0 is a successor apply Lemma 5 to a q' extending p with $\operatorname{dom}(q') = \beta - 1$, which exists by the choice of β_0. If β_0 is a limit ordinal pick an increasing sequence $\langle \alpha_i : i < \mu \rangle$ ($\mu < \lambda$) unbounded in β_0 (λ is singular, so $\operatorname{cf}(\beta_0) < \lambda$). Now play a game of length μ such that $p_0 = p$. Player I uses the strategy and Player II picks at stage $i + 1$ an extension of p_i with domain at least α_i. Such an extension exists as $\alpha_i < \beta_0$. Now $\langle p_i : i < \mu \rangle$ has some q extending all p_i's, so necessarily $\operatorname{dom}(p) \geq \beta_0$, contradicting the choice of β_0. □

Lemma 9. *If* G *is a generic filter for* $\mathbb{P}$*, then*:

(i) $\mathcal{S}_\lambda$ *holds in* $\mathbf{V}[G]$.

(ii) $\mathbf{V}$ *and* $\mathbf{V}[G]$ *share the same cardinals, power function and cofinalities.*

Proof. (i) Naturally we define in $\mathbf{V}[G]$ the sequence

$$\langle C^i_\alpha : \alpha < \lambda^+, i < \operatorname{cf}(\lambda) \rangle$$

as the union of all $p(\alpha)$'s for conditions p in G and α's in their domain. Clearly it is an $\mathcal{S}_\lambda$ sequence.

(ii) By Lemma 7 no subsets of size $< \lambda$ are added to $\mathbf{V}$ by G; as λ is singular, no subsets of size λ are added to $\mathbf{V}$. Therefore cardinals $\leq \lambda^+$ are not collapsed and cofinalities $\leq \lambda^+$ are not changed. Since $|\mathbb{P}| = 2^\lambda = \lambda^+$, it trivially satisfies the λ^{++}-cc. Hence cardinals and cofinalities above λ^+ are preserved. □

In $\mathbf{V}^{\mathbb{P}}$ we introduce a further forcing notion $\mathbb{R}$. Let

$$\langle C^i_\alpha : \alpha < \lambda^+, i < \operatorname{cf}(\lambda) \rangle$$

be the $\mathbb{P}$-generic $\mathbf{S}_\lambda$-sequence.

A *condition* $r \in \mathbb{R}$ is a closed bounded subset of λ^+ such that $\alpha \in r$ implies that for some $i < \operatorname{cf}(\lambda)$ the set C^i_α is unbounded in α.

$\mathbb{R}$ *is ordered* by end extensions.

The forcing notion $\mathbb{R}$ is designed to introduce a closed unbounded subset in λ^+, along which, for each α some C^i_α contains an unbounded subset of α. Such a c.u.b. is needed in order to construct for each $\mu < \lambda$ a partial order $\mathbb{Q}_\mu$ such that the iteration $\mathbb{P} * \mathbb{R} * \mathbb{Q}_\mu$ is μ-closed and forcing with $\mathbb{Q}_\mu$ (over $\mathbf{V}^{\mathbb{P}*\mathbb{R}}$) preserves stationary subsets of $S^{\lambda^+}_\mu = \{\delta < \lambda^+ : \operatorname{cf}(\delta) < \mu\}$.

The model in which our theorem is realized is $\mathbf{V}^{\mathbb{P}*\mathbb{R}}$, so we will study the properties of $\mathbb{P} * \mathbb{R}$ (rather than those of $\mathbb{R}$).

Let us work in the ground model $\mathbf{V}$. The iteration $\mathbb{P} * \mathbb{R}$ can be represented as the set of all pairs $\langle p, r \rangle$ such that $p \in \mathbb{P}$, and $p \Vdash$ "$r \in \mathbb{R}$". Note that as $\mathbb{P}$ does not introduce any new sets of size $\leq \lambda$, each member of $\mathbb{R}$ is in $\mathbf{V}$. It is easy to see that

> $p \Vdash$ "$r \in \mathbb{R}$" if and only if
>
> $r \subseteq \operatorname{dom}(p) + 1$ and for every $\alpha \in r$ there is some C^i_α in $p(\alpha)$ which is unbounded in α, and of course r is closed (as a subset of $\operatorname{dom}(p) + 1$).

Lemma 10. *The set*

$$\{\langle p, r \rangle : \langle p, r \rangle \in \mathbb{P} * \mathbb{R} \text{ and } \sup(r) = \operatorname{dom}(p)\}$$

is dense in $\mathbb{P} * \mathbb{R}$.

Proof. Given any $\langle p, r\rangle \in \mathbb{P} * \mathbb{R}$, define a one-level extension q of p as in the proof of Lemma 5. As $\beta(= \mathrm{dom}(p)+1) = \mathrm{dom}(q)$ is a member of each C^i_β, we may define $r' = r \cup \{\beta\}$ to get $\langle q, r'\rangle$ in $\mathbb{P} * \mathbb{R}$ above $\langle p, r\rangle$. $\square$

From now on let us assume that all the members of $\mathbb{P}*\mathbb{R}$ have this property (the second coordinate is a closed cofinal subset of the domain of the first).

Lemma 11. *For any $\langle p, r\rangle \in \mathbb{P} * \mathbb{R}$ and any $\alpha < \lambda^+$ there is a condition $\langle p', r'\rangle \geq \langle p, r\rangle$ such that $\alpha \in \mathrm{dom}(p') = \sup(r')$.*

Proof. This is an easy consequence of Lemmas 10 and 8. $\square$

Lemma 12. *The forcing notion $\mathbb{P}*\mathbb{R}$ is μ-strategically-closed for any regular $\mu < \lambda$.*

Proof. The strategy for Player I will be an adaptation of the strategy presented in the proof of Lemma 7. Let $\langle\langle p_i r_i\rangle i : i < \rho\rangle$ be the sequence played so far. We are going to define $\langle p_\rho, r_\rho\rangle$—the next condition picked by Player I.

We start with the successor stages.

For $\rho = 2$ we pick any one-level extension of $\langle p_1, r_1\rangle$.

For ρ successor bigger than 2, we modify the definition of the $C^i_{\gamma+1}$ from Lemma 7 by defining

$$C^i_{\gamma+1} = \begin{cases} \varnothing & \text{if } i < i_0, \\ \{\beta\} \cup C^i_\beta & \text{if } i_0 \leq i < j, \\ \{\gamma\} \cup C^i_\gamma & \text{if } j \leq i. \end{cases}$$

As $\gamma \in C^i_{\gamma+1}$ for $i \geq j$, we may define $r_\rho = r_{\rho-1} \cup \{\gamma\}$.

Now we are left with the limit stages.

Let γ be $\bigcup_{i<\rho} \mathrm{dom}(p_i)$ $(= \bigcup_{i<\rho} \sup(r_i))$. We repeat the definition of the $\mathbb{P}$ part of Lemma 7: $C^i_\gamma = \bigcup_{\alpha\in E_p} C^i_\alpha$ (the union of the C^i_α for all α's where $\alpha = \mathrm{dom}(p_i)$ for p_i's played by Player I). The $\mathbb{R}$ part can only be $r_p = \bigcup_{i<\rho} r_i \cup \{\gamma\}$.

We have to verify that indeed $\langle p_\rho, r_\rho\rangle \in \mathbb{P}*\mathbb{R}$. The only potential problem is that maybe there is no C^i_γ unbounded in γ. But, as $\mathrm{dom}(p_i) \in C^{i_0}_\alpha$ for $\alpha \in E_p$, where i is any even ordinal such that $\mathrm{dom}(p_i) < \alpha$, we get $E_p \subseteq C^{i_0}_\gamma$, so C^i_γ is unbounded in γ. $\square$

Lemma 13. *Forcing with $\mathbb{P} * \mathbb{R}$ does not add sets of size $\leq \lambda$ to the ground model, does not collapse cardinals or change cofinalities. It introduces an $\mathcal{S}_\lambda$ sequence $\langle C^i_\alpha : i < \mathrm{cf}(\lambda), \alpha < \lambda^+\rangle$ and a closed unbounded subset $C \subseteq \lambda^+$ such that for $\alpha \in C$ some C^i_α is unbounded in α.*

Proof. The proof is just a straightforward adaptation of the proof of Lemma 9. $\square$

The next step is, of course, to prove that in $\mathbf{V}^{\mathbb{P}*\mathbb{R}}$ every stationary subset of λ^+ reflects. Let $S \subseteq \lambda^+$ be stationary.

For a supercompact κ and an ordinal ρ if $\mathrm{cf}(\rho) > \kappa$, then every stationary subset of $S^\rho_\kappa = \{\alpha : \alpha < \rho, \mathrm{cf}(\alpha) < \kappa\}$ reflects. Working in $\mathbf{V}^{\mathbb{P}*\mathbb{R}}$ we define partial orders $\mathbb{Q}^{\lambda_i}_S = \mathbb{Q}^{\lambda_i}(S)$ for every λ_i (for $i < \mathrm{cf}(\lambda)$) and every stationary subset $S \subseteq \{\alpha < \lambda^+ : \mathrm{cf}(\alpha) < \lambda_i\}$.

Each $\mathbb{Q}^{\lambda_i}_S$ satisfies:

(i) $\mathbb{P} * \mathbb{R} * \mathbb{Q}^{\lambda_i}_S$ is a λ_i-directed closed forcing notion (in $\mathbf{V}$).

(ii) In $\mathbf{V}^{\mathbb{P}*\mathbb{R}*\mathbb{Q}^{\lambda_i}_S}$ the set S is a stationary subset of $(\lambda^+)^{\mathbf{V}^{\mathbb{P}*\mathbb{R}}}$ (= the ordinal that is λ^+ in $V^{\mathbb{P}*\mathbb{R}}$).

If such $\mathbb{Q}_S^{\lambda_i}$ exist, then, working in $\mathbf{V}^{\mathbb{P}*\mathbb{R}}$, for every stationary $S \subseteq \lambda^+$, pick $i < \mathrm{cf}(\lambda)$ such that the set $S_{\lambda_i} = \{\alpha \in S : \mathrm{cf}(\alpha) < \lambda_i\}$ is stationary. Then force with the appropriate $\mathbb{Q}_{S_{\lambda_i}}^{\lambda_i}$. In this forcing extension S_{λ_i} is stationary in $(\lambda^+)^{\mathbf{V}^{\mathbb{P}*\mathbb{R}}}$ and λ_i is a supercompact cardinal (as in $\mathbf{V}$, λ_i was an indestructible supercompact cardinal and $\mathbb{P} * \mathbb{R} * \mathbb{Q}_{S^{\lambda_i}}^{\lambda_i}$ is λ_i-directed-closed). It follows that

$$\mathbf{V}^{\mathbb{P}*\mathbb{R}*\mathbb{Q}^{\lambda_i}(S^{\lambda_i})} \models \text{“}S_{\lambda_i} \text{ reflects”},$$

i.e. for some $\alpha < (\lambda^+)^{\mathbf{V}^{\mathbb{P}*\mathbb{R}}}$, the intersection $S_{\lambda_i} \cap \alpha$ is stationary in α. It follows that in $\mathbf{V}^{\mathbb{P}*\mathbb{R}}$ the set $S_{\lambda_i} \cap \alpha$ is stationary in α and hence $S \cap \alpha$ is stationary. Therefore S reflects.

We are left with the task of constructing the $\mathbb{Q}_S^{\lambda_i}$'s.

Definition of the $\mathbb{Q}_S^{\lambda_i} = \mathbb{Q}^{\lambda_i}(S)$. We work in $\mathbf{V}^{\mathbb{P}*\mathbb{R}}$. Let $\langle C_\alpha^i : \alpha < \lambda^+, i < \mathrm{cf}(\lambda)\rangle$ be the $\mathcal{S}_\lambda$-sequence generated by the $\mathbb{P}$ generic set, and let C be the closed unbounded subset of λ^+ added by $\mathbb{R}$. For $i < \mathrm{cf}(\lambda)$ and a stationary set $S \subseteq \{\alpha < \lambda^+ : \mathrm{cf}(\alpha) < \lambda_i\}$, let j_0 be such that the set

$$S'_{j_0} = \{\alpha \in C \cap S : C_\alpha^{j_0} \text{ is unbounded in } \alpha\}$$

is stationary. (Recall that by the definition of C for each $\alpha \in S \cap C$ there is such a j_0.) Without loss of generality we may assume $S'_{j_0} = S$.

A condition $q \in \mathbb{Q}_S^{\lambda_i}$ is a bounded subset of λ^+ such that $\mathrm{otp}(q) < \lambda_i$.
The order on $\mathbb{Q}_S^{\lambda_i}$ is that of end-extensions.

Note that the only role of S is determining j_0.

Lemma 14. *For every S and every λ_i the partial order $\mathbb{Q}_S^{\lambda_i}$ is (less than) λ_i-closed.*

Proof. This is trivial as the definition of a condition is closed under unions of size $< \lambda_i$. □

Definition 15. *A condition* $\langle p, r, q\rangle \in \mathbb{P} * \mathbb{R} * \mathbb{Q}_S^{\lambda_i}$ *is* leveled *if* $\mathrm{dom}(p) = \sup(r) = \sup(q)$.

Lemma 16. *The following holds in the ground model* $\mathbf{V}$:

(a) *The set of leveled conditions is dense in $\mathbb{P} * \mathbb{R} * \mathbb{Q}_S^{\lambda_i}$.*
(b) *The set of leveled conditions of $\mathbb{P} * \mathbb{R} * \mathbb{Q}_S^{\lambda_i}$ is λ_i-closed.*

Proof. We may assume that the minimal condition of $\mathbb{P} * \mathbb{R}$ forces that S is a stationary subset of $\{\alpha < \lambda^+ : \mathrm{cf}(\alpha) < \lambda_i\}$ and decides the value of j_0.

(a) By Lemma 10 we may assume $\mathrm{dom}(p) = \sup(r)$ as q is forced by $\langle p, r\rangle$ to be a member of $Q_S^{\lambda_i}$ and $\sup(q)$ cannot exceed $\mathrm{dom}(p)$. Denote $\mathrm{dom}(p) = \delta + 1$ and $\sup(q) = \gamma$. Let i^* be the first i such that $\gamma \in C_\delta^{i^*}$, and let p' be the one-level extension of p defined by $p'(\delta+1) = \langle C_{\delta+1}^i : i < \mathrm{cf}(\lambda)\rangle$, where

$$C_{\delta+1}^i = \begin{cases} \varnothing & \text{if } i < j_0, \\ C_\gamma^i & \text{if } j_0 \le i < i^*, \\ C_\delta^i \cup \{\delta+1\} & \text{if } i^* < i < \mathrm{cf}(\lambda). \end{cases}$$

Let r' be $r \cup \{\delta+1\}$ and $q' = q \cup \{\delta+1\}$. It should be clear that $\langle p', r', q'\rangle \in \mathbb{P} * \mathbb{R} * \mathbb{Q}_S^{\lambda_i}$ is leveled.

(b) We can restrict ourselves to the set of leveled conditions. First we note that in $\mathbb{P} * \mathbb{R} * \mathbb{Q}_S^{\lambda_i}$, if two conditions are compatible, then they are comparable (i.e.

if $\langle p, r, q\rangle$ and $\langle p', r', q'\rangle$ have a common extension, then one of them is above the other). Therefore, the notions of being μ-closed and μ-directed-closed coincide for this partial order.

Let $\langle\langle p_j, r_j, q_j\rangle : j < \rho < \lambda_i\rangle$ be an increasing sequence of leveled conditions. Let $\delta = \sup\{\mathrm{dom}(p_i) : i < \rho\}$ and let p_ρ be $\bigcup_{i<\rho} p_i \hat{}\, p_\rho(\delta)$, where $p_\rho(\delta)$ is the sequence $\langle C^i_\delta : i < \mathrm{cf}(\lambda)\rangle$ defined by

$$C^i_\delta = \begin{cases} \varnothing & \text{if } i < j_0, \\ \bigcup\{C^i_\alpha : \alpha \in \bigcup_{i<\rho} q_i\} & \text{if } j_0 \le i < \mathrm{cf}(\lambda). \end{cases}$$

Let $r_\rho = \bigcup_{i<\rho} r_i \cup \{\delta\}$ and $q = \bigcup_{i<\rho} q_i$. Applying Lemma 6, it is straightforward to check that $\langle p_\rho, r_\rho, q_\rho\rangle$ is a condition extending each $\langle p_i, r_i, q_i\rangle$. □

Lemma 17. *If in $\mathbf{V}^{\mathbb{P}*\mathbb{R}}$, S is a stationary subset of*

$$\{\alpha < \lambda^+ : \mathrm{cf}(\alpha) < \lambda_i,\ C^i_\alpha \text{ is unbounded in } \alpha\},$$

then in $\mathbf{V}^{\mathbb{P}*\mathbb{R}*\mathbb{Q}^{\lambda_i}_S}$, the set S is stationary in $(\lambda^+)^{\mathbf{V}^{\mathbb{P}*\mathbb{R}}}$.

Proof. We work in $V^{\mathbb{P}*\mathbb{R}}$. Assume,[2] by way of contradiction, that $q_0 \in \mathbb{Q}^{\lambda_i}_S$ and a $\mathbb{Q}^{\lambda_i}_S$-name $\dot\tau$ are such that q_0 forces that $\dot\tau$ is a closed unbounded subset of $(\lambda^+)^{\mathbf{V}^{\mathbb{P}*\mathbb{R}}}$ disjoint from S. For $\alpha < \lambda^+$ let

$$T_\alpha = \{t : t \subseteq C^{j_0}_\alpha \cup \{\alpha\} \& \alpha \in t\}.$$

Note that $\beta \in t \in T_\alpha \Rightarrow t \cap (\alpha+1) \in T_\beta$. We choose by induction on $\alpha < \lambda^+, T'_\alpha \subseteq T_\alpha$ and for every $t \in T'_\alpha$, a condition $q_t \in \mathbb{Q}^{\lambda_i}_S$ and an ordinal ζ_t such that

(a) $\beta \in t \Rightarrow q_{t\cap(\beta+1)} \le q_t$,
(b) $q_t \Vdash$ "$\zeta_t \in \dot\tau$", $\max(t) < \zeta_t < \lambda^+$,
(c) $T'_\alpha = \{t \in T_\alpha : (\forall \beta \in t)(t \cap (\beta+1) \in T'_\beta \& \sup(q_{t\cap(\beta+1)}) < \alpha)\}$.

Note that $|T_\alpha| \le \lambda$, so for some closed unbounded $E \subseteq \lambda^+$ we have

$$\alpha < \delta \in E \& t \in T'_\alpha \Rightarrow \sup(q_t) < \delta.$$

Take $\delta \in S \cap E$ and choose $t \subseteq C^{j_0}_\delta$ unbounded in δ of order type $\mathrm{cf}(\delta) < \lambda_i$ such that

$$\alpha \in t \& \beta \in t \& \alpha < \beta \Rightarrow E \cap (\alpha, \beta) \ne \varnothing.$$

Easily

$$\alpha \in t \Rightarrow t \cap (\alpha+1) \in T'_\alpha, \qquad q = \bigcup_{\alpha\in t} q_{t\cap(\alpha+1)} \quad \text{is in } \mathbb{Q}^{\lambda_i}_S,$$

and the condition q is above each $q_{t\cap(\alpha+1)}$. Hence $\alpha \in t$ implies that $q \Vdash$ "$\zeta_t \in \dot\tau$" and $\alpha < \zeta_\alpha < \min(t \setminus (\alpha+1))$. Thus $q \Vdash$ "$\delta \in \dot\tau \cap S$", contradicting the assumption that $q_0 \Vdash$ "$S \cap \dot\tau = \varnothing$". □

---

[2] Actually the proof is by [Sh:108], as in $\mathbf{V}^{\mathbb{P}*\mathbb{R}}$, $\lambda^+ \in I[\lambda^+]$, then $\mathbb{Q}^{\lambda_i}_S$ (really it is Levy (λ_i, λ^+)) is λ_i-closed hence preserves stationarity of $S \subseteq \{\delta < \lambda^+ : \mathrm{cf}(\delta) < \lambda_i\}$. But we give specific proof.

4. A generalization

Our main theorem is stated in terms of the existence of some cardinal λ with the desired properties. Using results from [BD86] we get a generalization of the theorem to every singular λ.

Theorem 18 (Ben-David)**.** *If ZFC is consistent with the existence of a proper class of supercompact cardinals, then it is consistent with the statement*

> *For every regular cardinal $\mu < \lambda$ every stationary subset of $S^\mu_\lambda = \{\delta < \lambda : \mathrm{cf}(\delta) < \mu\}$ reflects and reflection of subsets of S^λ_μ is retained after forcing with μ-directed-closed forcing notions.*

Proof. This is the content of Theorems 4.1, 4.5 and Remark 4.7 of [BD86]. $\square$

Theorem 19. *For every singular cardinal λ, if ZFC is consistent with the existence of class many supercompact cardinals, then it cannot be proved that $(\aleph_1, \aleph_0) \to (\lambda^+, \lambda)$ implies the existence of a non-reflecting stationary subset of λ^+.*

Proof. Just note that the conclusion of Theorem 18 is all we need to prove the main theorem. See more in [Sh:351]. $\square$

References

[BD86] S. Ben David, *Full reflection of stationary sets*, Abstracts Amer. Math. Soc. **7** (1986), 200.

[BdSh:203] S. Ben David and S. Shelah, *Souslin trees and successors of singular cardinals*, Ann. Pure Appl. Logic **30** (1985), 207–217. MR **87h:**03078

[BM86] S. Ben David and M. Magidor, *The weak $\square^*_\lambda$ is really weaker than the full $\square_\lambda$*, J. Symbolic Logic **51** (1986), 1029–1033. MR **88a:**03117

[BdSh:236] S. Ben David and S. Shelah, *Non-special Aronszajn trees on $\aleph_{\omega+1}$*, Israel J. Math. **53** (1986), 93–96. MR **87k:**03051

[BD86] S. Ben David, *A Laver-type indestructability for accessible cardinals*, Logic Colloquium '86 (F. R. Drake and J. K. Truss, eds.), North-Holland, Amsterdam, 1988, pp. 9–19. MR **89e:**03093

[E80] P. Eklof, *Set theoretic methods in homological algebra and abelian groups*, Presses Univ. Montréal, Montréal, 1980. MR **81j:**20004

[F83] W. G. Fleissner, *If all Moore spaces are metrizable, then there is an inner model with a measurable cardinal*, Trans. Amer. Math. Soc. **273** (1982), 365–373. MR **84h:**03118

[G76] J. Gregory, *Higher Souslin trees and the generalized continuum hypothesis*, J. Symbolic Logic **41** (1976), 663–671. MR **58:**5208

[J72] R. B. Jensen, *The fine structure of the constructable hierarchy*, Ann. Math. Logic **4** (1972), 229–308. MR **46:**8834

[K78] K. Kunen, *Saturated Ideals*, J. Symbolic Logic **43** (1978), 65–76. MR **80a:**03068

[L78] R. Laver, *Making the supercompactness of κ indestructible under κ-directed closed forcing*, Israel J. Math. **29** (1978), 385–388. MR **57:**12226

[M82] M. Magidor, *Reflecting stationary sets*, J. Symbolic Logic **47** (1982), 755–771. MR **84f:**03046

[Mi72] W. Mitchell, *Aronszajn trees and the independence of the transfer property*, Ann. Math. Logic **5** (1972), 21–46. MR **47:**1612

[Sh:44] S. Shelah, *Infinite abelian groups, Whitehead problem and some constructions*, Israel J. Math. **18** (1974), 243–256. MR **50:**9582

[Sh:52] ———, *A compactness theorem for singular cardinals for free algebras. Whitehead problem and transversals*, Israel J. Math. **21** (1975), 329–349. MR **52:**10410

[Sh:108] ———, *On the successors of singular cardinals*, Logic Colloquium '78 (M. Boffa, D. Van Dalen, and K. McAloon, eds.), North-Holland, Amsterdam, 1979, pp. 357–380. MR **82d:**03079

[Sh:269] ______, "*Gap* 1" *two cardinal principle and omitting type theorems for* $\mathcal{L}(Q)$, Israel J. Math. **65** (1989), 133–152. MR **91a:**03083

[Sh:351] ______, *Reflecting stationary sets and successors of singular cardinals*, Arch. Math. Logic **31** (1991), 25–53. MR **93h:**03072

Department of Computer Science, Technion-Israel Institute of Technology, Haifa, Israel

E-mail address: `shai@cs.technion.ac.il`

Department of Mathematics, The Hebrew University, Jerusalem, Israel