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EMBEDDING OF A BANACH ALGEBRA A INTO $L(A')$

ETIENNE DESQUITH

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Abstract. Given a Banach space X, we have shown in 1994 that a product can be defined on it in such a way that the resulting Banach algebra is isomorphic to a compact subalgebra of the algebra $L(X')$ of all bounded linear operators on the topological dual X' of X. Our purpose here is to prove that, more generally, any Banach algebra A admitting a *left approximate identity*, is isomorphic to a subalgebra of $L(A')$, the isomorphism being isometric, provided the approximate identity is bounded by 1. As a consequence, we get a factorization through $L(A')$, of the elements in A'.

An abstract C^*-algebra is isometrically isomorphic to a uniformly closed subalgebra of $L(H)$, the algebra of all bounded linear operators on some Hilbert space H. When A is no longer a C^*-algebra, but some Banach algebra with a left approximate identity, we still get an isomorphism from A into an algebra of bounded linear operators on some Banach space E which is no longer a Hilbert space. More precisely, E is the topological dual A' of A. When the approximate identity is a subset of the unit sphere of A, we get a similar identification of A as in the case of a C^*-algebra, in the following way:

Any Banach algebra A with a left approximate identity bounded by 1 *is isometrically isomorphic to a uniformly closed subalgebra of* $L(A')$.

When A is a commutative Banach *-algebra, then the isomorphism is an involution-preserving map.

We mainly derive from the above identification of A with a subalgebra of $L(A')$, a particular factorization of the elements of A', through $L(A')$.

Let A and B be Banach spaces each of which is endowed with a linear involution denoted by "$*$" and "$\bullet$" respectively. We consider:

(1) The linear involution "$\otimes$" defined on the space $L(A, B)$ by:

$$P^{\otimes}(a) = P(a^*)^{\bullet} \qquad (P \in L(A,B); a \in A). \tag{1}$$

(2) The linear involution "$\ltimes$" defined on the space A' by:

$$T^{\ltimes}(x) = \overline{T(x^*)} \qquad (x \in A; T \in A') \tag{2}$$

($T^{\ltimes}(x) = \overline{T(x)}$, if A does not have an involution, where the bar denotes complex conjugation).

The following is the main result of the paper.

Theorem 1. *Let A be a Banach algebra with a left approximate identity $(u_\alpha)_{\alpha\in I}$ bounded by* 1. *Then*

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(i) *There exists an isometric isomorphism* σ *from* A *onto a uniformly closed subalgebra of* $L(A')$.

(ii) *If* A *is a commutative Banach* $*$*-algebra, then the isometry* σ *is* $*$*-map in the following sense:* $\sigma(x^*) = \sigma(x)^{\otimes}, \forall x \in A$.

Proof. For all $x, y, \in A$, and all $T \in A'$, we set:

$$(\sigma(x)(T))(y) = T(yx). \tag{3}$$

Then easy computations show that the map $x \mapsto \sigma(x)$ from A into $L(A')$ induced from relation (3) above is a homomorphism. Moreover, σ is one-to-one as follows: if $x, y \in A$ are such that $\sigma(x) = \sigma(y)$, then for all $z \in A$ and $T \in A'$, we have $T(zx) = T(zy)$. But since A admits a left approximate identity $(u_\alpha)_{\alpha \in I}$, we get

$$T(u_\alpha x) = T(u_\alpha y) \qquad (\alpha \in I, T \in A'), \tag{4}$$

that is,

$$\begin{aligned} \|T(x-y)\| &= \|T(u_\alpha(x-y)) - T(x-y)\| \\ &\leq \|T\|\,\|u_\alpha(x-y) - (x-y)\|, \end{aligned}$$

from which it follows that

$$T(x) = T(y) \quad \forall T \in A', \tag{5}$$

and therefore, $x = y$.

Moreover, for all $x \in A$, it is easily checked that

$$\|\sigma(x)\| \leq \|x\|. \tag{6}$$

Conversely, if A_1 (resp. A'_1) denotes the closed unit ball of A (resp. A'), then given $x \in A$, we have, for all $\alpha \in I$:

$$\begin{aligned} \|u_\alpha x\| &\leq \operatorname{Sup}\{\|yx\|; y \in A_1\} \\ &\leq \operatorname{Sup}\{\operatorname{Sup}\{|T(yx)|; T \in A'_1\}; y \in A_1\} \\ &\leq \operatorname{Sup}\{\operatorname{Sup}\{|(\sigma(x)T)(y)|; T \in A'_1\}; y \in A_1\}. \end{aligned}$$

It follows, when α tends to infinity, that

$$\|x\| \leq \|\sigma(x)\|, \tag{7}$$

which, together with relation (6), shows that σ is isometric. The range $\sigma(A)$ of σ is clearly closed under the uniform norm of $L(A')$.

(ii) If A is in addition a commutative $*$-algebra, then σ is an involution-preserving map, for if $x, y \in A$ and $T \in A'$, then

$$\begin{aligned} (\sigma(x^*)(T))(y) &= T(yx^*) = T(x^*y) \quad (A \text{ is commutative}) \\ &= \overline{T^{\ltimes}(y^*x)} = \overline{(\sigma(x)(T^{\ltimes}))(y^*)} \\ &= (\sigma(x)(T^{\ltimes}))^{\ltimes}(y) = ([\sigma(x)]^{\otimes}(T))(y), \end{aligned}$$

from which it follows that:

$$\sigma(x^*) = [\sigma(x)]^{\otimes}, \tag{8}$$

and the proof is complete. □

Corollary 2. *If* A *is a commutative* C^**-algebra, then* $\sigma(A)$ *endowed with the involution* $\otimes$ *is a commutative* C^**-algebra.*

Proof. Each C^*-algebra admits a two-sided approximate identity bounded by 1. According to Theorem 1 (ii) above, $\sigma(A)$ is a commutative $\otimes$-sub-algebra of $L(A')$ satisfying, for all $x \in A$:

$$\|\sigma(x)^{\otimes}\sigma(x)\| = \|\sigma(x^*x)\| = \|x^*x\| = \|x^2\| = \|\sigma(x)\|^2. \qquad \square$$

Corollary 3. *Let G be a topological locally compact group and $C_0(G)$ the algebra of all continuous complex functions on G vanishing at infinity.*

(i) *If $M^1(G)$ denotes the space of all bounded Radon measures on G, then $L(M^1(G))$ contains an isometric image of $C_0(G)$.*

(ii) *If G is abelian, then $L^1(G)$ is isometrically isomorphic to a subalgebra of $L(L^\infty(G))$.*

Proof. (i) $A = C_0(G)$ endowed with the sup-norm and the involution defined by complex conjugation is a commutative C^*-algebra. Hence according to Theorem 1 and Corollary 2, A may be identified with a commutative C^*-algebra in $L(A')$. But the Riesz representation theorem ensures that A' is isometrically isomorphic to $M^1(G)$.

(ii) If G is abelian, the group algebra $L^1(G)$ endowed with the usual convolution of functions is a commutative Banach $*$-algebra with a two-sided approximate identity bounded by 1. Hence Theorem 1 holds, and the result follows. $\square$

We shall need the following lemma for the next result.

Lemma 4. *Let X, Y, Z be Banach spaces, and P, Q be linear operators from X into Y and from X into Z respectively, such that* $\mathrm{Ker}(P)$ *and* $\mathrm{Ker}(Q)$ *satisfy* $\mathrm{Ker}(P) \subset \mathrm{Ker}(Q)$*. Then there exists a linear map S from the range $R(P)$ of P into Z satisfying $Q = SP$.*

Proof. Straightforward. $\square$

We have the following particular decomposition of the elements of A':

Theorem 5. *Let A be a Banach algebra with a left approximate identity bounded by* 1*. Then, each $T \in A'$ admits the following factorization:*

$$T = \Phi\Theta\sigma, \tag{9}$$

where

σ is the isometry in Theorem 1*,*
Θ is a continuous linear map from $\sigma(A) \subset L(A')$ into A', and
Φ is an element in the bidual A'' of A.

Proof. Let $(u_\alpha)_{\alpha\in I}$ be a left approximate identity. For each $T \in A'$, we define the map Ψ_T as follows:

$$x \mapsto \psi_T(x) = \sigma(x)(T) \qquad (x \in A). \tag{10}$$

ψ_T is well-defined, linear and continuous ($\|\psi_T\| \le \|T\|$), and takes its values in A'. Moreover, using the existence of a (bounded) left approximate identity in A, one shows that

$$\mathrm{Ker}(\psi_T) \subset \mathrm{Ker}(T) \tag{11}$$

$[(\sigma(x)(T))(\cdot) = T(\cdot x)]$. But then, according to Lemma 4, there exists a linear map φ from the range $\psi_T(A)$ of ψ_T onto $\mathbb{C}$ such that $T = \varphi\psi_T$. φ is continuous on

$\psi_T(A)$. Indeed, let $\varepsilon > 0$ be given and let $x, y \in A$ be such that $\|\psi_T(x) - \psi_T(y)\| < \varepsilon' = \frac{\varepsilon}{2(1+\|T\|)}$. This implies, for all $z \in A_1, |T(z(x-y))| < \varepsilon'$.

It follows that, for each element u_α of a left approximate identity $(u_\alpha)_{\alpha\in I}$ bounded by 1, the inequality $|T(u_\alpha(x-y))| < \varepsilon'$ holds, and therefore:

$$\begin{aligned}|T(x-y)| &\leq |T(u_\alpha(x-y))| + |T(x-y) - T(u_\alpha(x-y))| \\ &< \varepsilon' + \|T\|\, \|(x-y) - u_\alpha(x-y)\| \\ &< (1+\|T\|)\,\varepsilon' \quad \left(\text{since } \lim_{\alpha\to+\infty}[(x-y) - u_\alpha(x-y)] = 0\right) \\ &< (1-\|T\|)\frac{\varepsilon}{2(1+\|T\|)} = \frac{\varepsilon}{2}.\end{aligned}$$

Hence $|\varphi(\psi_T(x)) - \varphi(\psi_T(y))| < \frac{\varepsilon}{2}$, and φ is continuous on $\psi_T(A)$. Then, the Hahn-Banach continuous extension theorem yields a continuous linear map Φ from A' onto $\mathbb{C}$ ($\Phi \in A''$) which coincides with φ on $\psi_T(A)$ and satisfies:

$$T = \Phi\psi_T. \tag{12}$$

Furthermore, $\operatorname{Ker}(\sigma) = \{0\} \subset \operatorname{Ker}(\psi_T)$, and once again Lemma 4 yields a linear map Θ from $\sigma(A)$ onto $\psi_T(A)$, which is continuous from the isometric nature of σ, and such that:

$$\psi_T = \Theta\sigma. \tag{13}$$

Hence T finally fulfills $T = \Phi\Theta\sigma$, which is the desired factorization. □

Now, let X be a Banach space and F a continuous linear form on X. If for $x, y \in X$ one defines:

$$xy = F(x)y, \tag{14}$$

then X, endowed with its Banach space norm together with the above product, becomes a complete normed algebra $A(X,F)$, which is a Banach algebra if $\|F\| \leq 1$.

If $f \in X'$, the nonzero vector $x \in X$ is called a *maximal element* of f, if $f(x) = \|f\|\,\|x\|$ (cf. [5], Definition 15.2.1, page 337).

We have proved in [3] the following theorem and its corollary:

Theorem 6. *Let X be a Banach space. If F is in the unit sphere of X' and admits a maximal element $u \in X$, then $A(X,F)$ is isometrically isomorphic to a uniformly closed subalgebra of the algebra $\mathcal{K}(X')$ of all compact operators on X' (cf.* [3], *Théorème* 2.19).

Corollary 7. *If X is a reflexive Banach space or the dual space Y' of a Banach space Y, and if F (resp. Φ) is an element of the unit sphere of X' (resp. Y''), then $A(X,F)$ (resp. $A(Y',\Phi)$) is isometrically isomorphic to a uniformly closed subalgebra of $\mathcal{K}(X')$ (resp. $\mathcal{K}(Y'')$) (cf.* [3], *Corollaire* 2.20). □

The isomorphism ρ in the above theorem satisfies

$$(\rho(x)(T))(y) = T(yx) = F(y)T(x), \qquad x, y \in X. \tag{15}$$

Moreover, the range, say $A_\rho(X,F)$ of ρ in $\mathcal{K}(X')$, consists of rank-one operators on X'. Namely,

$$\rho(x)(T) = T(x)F = \hat{x}(T)F, \qquad x \in X, T \in X', \tag{16}$$

where $x \mapsto \hat{x}$ denotes the natural linear isometry of X into X''.

Remark 8. It turns out that Theorem 6 above is a particular case of Theorem 1, according to the following:

Proposition 9. *The Banach algebra $A(X,F)$ in Theorem 6 admits infinitely many left approximate identities bounded by* 1. *Moreover precisely, $(u_\alpha)_{\alpha\in I}$ is a left approximate identity for $A(X,F)$ if and only if one has:*

$$\lim_{\alpha\to+\infty} F(u_\alpha) = 1. \tag{17}$$

In particular, if $u \in X$ is such that $F(u) = 1$, then the sequence $(u_n)_{n\geq 1}$ such that $u_n = z_n u$, where $(z_n)_{n\geq 1}$ is any complex sequence in the unit disk ($\|z_n\| \leq 1; n \geq 1$) satisfying $\lim_{n\to+\infty} z_n = 1$, is a countable left approximate identity for $A(X,F)$, bounded by 1.

Proof. Straightforward. □

We close the sequel with the following two properties of $A_\rho(X,F)$:

Proposition 10. (i) *For all $x \in X$ and $P \in L(X')$, the following condition holds:*

$$P \circ \rho(x)(\cdot) = \hat{x}(\cdot)P(F). \tag{18}$$

(ii) *For all $x \in X$ and $P \in L(X')$, there exists, for each $T \in X'$, some $z_{x,T,P} \in X$* (*depending on x, P and T*) *such that*:

$$(\rho(x) \circ P)|_{\mathbb{C}T} = \rho(z_{x,T,P}), \tag{19}$$

where $(\rho(x)\circ P)|_{\mathbb{C}T}$ stands for the restriction of $(\rho(x)\circ P)$ to the one-dimensional subspace $\mathbb{C}T$ of X'.

Proof. (i) is obvious.

(ii) Let $x \in X$ and $P \in L(X')$. Then, for each $T \in X'$ and $\lambda \in \mathbb{C}$, we have $(\rho(x)\circ P)(\lambda T) = \lambda P(T)(x)F$. But we may choose $z_{x,T,P} \in X$ such that $P(T)(x) = T(z_{x,T,P})$, and it follows that

$$\begin{aligned}(\rho(x)\circ P)(\lambda T) &= \lambda T(z_{x,T,P})F = \lambda \hat{z}_{x,T,P}(T)F \\ &= \lambda\rho(z_{x,T,P})(T) = \rho(z_{x,T,P})(\lambda T). \qquad \square\end{aligned}$$

Corollary 11. *$A_\rho(X,F)$ is a proper subspace of $\mathcal{K}(X')$.*

Proof. $A_\rho(X,F)$ is clearly not reduced to $\{0\}$. Since $\mathcal{K}(X')$ is a two-sided ideal of $L(X')$, the assumption $A_\rho(X,F) = \mathcal{K}(X')$ would imply that, given $x \in X$ and $P \in L(X')$, there exists some $u_{x,P} \in X$ satisfying:

$$P \circ \rho(x) = \rho(u_{x,P}). \tag{20}$$

Let $\mathrm{id}_{L(X')}$ denote the identity map on $L(X')$. We may assume that $P \notin \mathbb{C}\,\mathrm{id}_{L(X')}$ in relation (20) above. According to relation (15), we get from relation (20) that

$$T(x)P(F)(y) = T(u_{x,P})F(y) \tag{21}$$

for all $y \in X$ and for some nonzero T in X'.

Now, for all $x \in X$ and for any y in the kernel $\mathrm{Ker}(F)$ of F, relation (21) above implies:

$$0 = T(x)P(F)(y) \qquad (x \in X, P \in L(X')),$$

and hence $y \in \mathrm{Ker}(P(F))$. Therefore, $\mathrm{Ker}(F) \subset \mathrm{Ker}(P(F))$ and the following holds:

$$P(F) = \lambda_P F, \qquad \lambda_P \in \mathbb{C}. \tag{22}$$

But this forces P to satisfy $P \in \mathbb{C}\,\mathrm{id}_{L(X')}$, in contradiction with the choice of P, and the conclusion follows. $\square$

Remark 12. In the proof of Corollary 11, we have in fact showed that $A_\rho(X, F)$ cannot be a left ideal of $L(X')$. Whether $A_\rho(X, F)$ is a right ideal of $L(X')$ or not, is a question to which we do not yet have an answer.

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Institut de Recherches Mathématiques (IRMA), 08 BP 2030 Abidjan 08, Cote D'Ivoire