

## POLYNOMIALLY BOUNDED OPERATORS AND EXT GROUPS

SARAH H. FERGUSON

(Communicated by Palle E. T. Jorgensen)

ABSTRACT. In this paper, we consider the Ext functor in the category of Hilbert modules over the disk algebra. We characterize the group  $\text{Ext}_{A(D)}(K, H)$  as a quotient of operators and explicitly calculate  $\text{Ext}_{A(D)}(K, H^2)$ , where  $K$  is a weighted Hardy space. We then use our results to give a simple proof of a result due to Bourgain.

### 1. INTRODUCTION

In 1974, Foias and Williams studied the class of  $2 \times 2$  operator matrices of the form below, [5]. Although their paper was never published, the main results appear in [3]. Foias and Williams conjectured that an operator of the form

$$R_f = \begin{pmatrix} S^* & \Gamma_f \\ 0 & S \end{pmatrix},$$

where  $S$  is the forward shift on  $\ell^2$  and  $\Gamma_f$  is the Hankel matrix with symbol  $f$ , is a counterexample to Halmos' famous problem: *Is every polynomially bounded operator similar to a contraction?*

What Foias and Williams proved was that  $R_f$  is similar to a contraction if and only if there is a bounded solution to the commutator equation  $\Gamma_f = S^*X - XS$ . This means that  $R_f$  is similar to a contraction if and only if  $R_f$  is similar to  $S^* \oplus S$  via a similarity of the form  $\begin{pmatrix} I & X \\ 0 & I \end{pmatrix}$ . By solving the commutator equation above, one sees that all solutions have the form  $X = \Gamma_g - \Gamma_f DS^*$ , where  $g \in H^2$  and  $D$  is the diagonal matrix  $\text{diag}(i+1)_{i \geq 0}$ . Paulsen observed that if  $X$  is a solution to the commutator equation, then  $-X^t$  is a solution as well. Hence

$$Y = \frac{X - X^t}{2} = \left( \frac{(i-j)}{2} \hat{f}(i+j-1) \right)_{i,j \geq 0}$$

is a bounded solution. Here  $\hat{f}(n)$  is the  $n^{\text{th}}$  Fourier coefficient of  $f$  and  $\hat{f}(-1) = 0$ . It follows that  $R_f$  is similar to a contraction if and only if the matrix  $Y$  is bounded on  $\ell^2$ .

Several other people have studied this operator including Peller who, in [6], proved that  $R_f$  is power bounded if and only if  $f'$  is in the Bloch class. He also

---

Received by the editors March 13, 1995.

1991 *Mathematics Subject Classification*. Primary 47B38; Secondary 18G15.

*Key words and phrases*. Polynomially bounded operator, Ext groups, reproducing kernel Hilbert space, weighted Hardy space, BMOA.

showed that if  $f'$  is BMOA, then  $R_f$  is polynomially bounded. Bourgain showed in [1] that if  $f'$  is BMOA, then  $R_f$  is similar to a contraction.

An operator  $T$  is polynomially bounded on  $H$  if and only if the map  $p \mapsto p(T)$  defined on polynomials extends to a representation of the disk algebra,  $\mathbf{A}(\mathbf{D})$ , on  $H$ . In terms of Hilbert modules, this means that the map  $(p, h) \mapsto p(T)h$  extends to a Hilbert  $\mathbf{A}(\mathbf{D})$ -module action on  $H$ . The first systematic study of Hilbert modules was done by Douglas and Paulsen in [4]. Carlson and Clark were the first to study the Ext functor in this category [2],[3].

In this paper we give a concrete characterization of  $\text{Ext}_{\mathcal{A}}(K, H)$  as a quotient of operators and use this together with a result from [2] to calculate the groups  $\text{Ext}_{\mathcal{A}(D)}(K, H^2)$  for a large class of Hilbert modules  $K$ . We then show how these results can be used to give an alternative proof of Bourgain's result mentioned above.

### 2. HOMOLOGICAL PRELIMINARIES

A Hilbert module  $H$  over a function algebra  $\mathcal{A}$  is a Hilbert space together with a bounded, unital homomorphism  $\pi : \mathcal{A} \rightarrow \mathcal{L}(H)$ . Such a map is called a representation of the algebra  $\mathcal{A}$  on  $H$ . Given a representation  $\pi$ , one defines the module action on  $H$  by  $a \cdot h = \pi(a)h$ . It is easy to see that every Hilbert module action arises this way. In fact, if  $\rho : \mathcal{A} \times H \rightarrow H$  defines a bounded module action on  $H$  then  $\pi(a)h \equiv \rho(a, h)$  defines a representation of  $\mathcal{A}$  on  $H$ .

Given two Hilbert  $\mathcal{A}$ -modules,  $(H, \pi_1)$  and  $(K, \pi_2)$ , an operator  $T \in \mathcal{L}(H, K)$  is called a Hilbert module map if  $T\pi_1(a) = \pi_2(a)T$  for all  $a \in \mathcal{A}$ .  $\text{Ext}_{\mathcal{A}}(K, H)$  is defined to be the collection of equivalence classes of short exact sequences of the form

$$(1) \quad 0 \rightarrow H \xrightarrow{\alpha} J \xrightarrow{\beta} K \rightarrow 0.$$

Here  $J$  is a Hilbert  $\mathcal{A}$ -module,  $\alpha$  and  $\beta$  are Hilbert module maps and exactness means that  $\alpha$  is 1-1,  $\beta$  is onto and the range of  $\alpha$  is equal to the kernel of  $\beta$ . We call two such sequences equivalent if there exists a Hilbert module map between the two middle modules such that the following diagram commutes:

$$\begin{array}{ccccccccc} 0 & \rightarrow & H & \xrightarrow{\alpha} & J & \xrightarrow{\beta} & K & \rightarrow & 0 \\ & & \parallel & & \downarrow & & \parallel & & \\ 0 & \rightarrow & H & \xrightarrow{\alpha'} & J' & \xrightarrow{\beta'} & K & \rightarrow & 0 \end{array}$$

In this category, every short exact sequence is equivalent to one of the form

$$0 \rightarrow H \xrightarrow{\iota} H \oplus K \xrightarrow{P} K \rightarrow 0.$$

Here the middle module is the Hilbert space direct sum (with an appropriately defined module action),  $\iota$  is the isometric inclusion and  $P$  is the orthogonal projection onto  $K$ . To see this fact, note from (1) we have  $\alpha(H) = \text{kernel}(\beta)$  so that  $\alpha(H)$  is closed in  $J$ . As a Hilbert space then,  $J = \alpha(H) \oplus \alpha(H)^\perp$ . Since the restriction of  $\beta$  to  $\alpha(H)^\perp$  maps 1-1 and onto  $K$ ,  $\beta$  has a right inverse  $T$ . Now define  $S \in \mathcal{L}(H \oplus K, J)$  by  $S(h + k) = \alpha(h) + T(k)$ . If  $\pi$  is the representation of  $\mathcal{A}$  on  $J$ , then  $\tilde{\pi}(a) \equiv S^{-1}\pi(a)S$  defines a representation of  $\mathcal{A}$  on  $H \oplus K$  such that the

following diagram commutes:

$$\begin{array}{ccccccccc}
 0 & \rightarrow & H & \xrightarrow{\iota} & H \oplus K & \xrightarrow{P} & K & \rightarrow & 0 \\
 & & \parallel & & \downarrow s & & \parallel & & \\
 0 & \rightarrow & H & \xrightarrow{\alpha} & J & \xrightarrow{\beta} & K & \rightarrow & 0
 \end{array}$$

If the sequence  $0 \rightarrow H \xrightarrow{\iota} H \oplus K \xrightarrow{P} K \rightarrow 0$  is exact and  $\pi$  is the representation on  $H \oplus K$ , then for each  $a \in \mathcal{A}$  we get the decomposition

$$\pi(a) = \begin{pmatrix} \pi_1(a) & \delta(a) \\ 0 & \pi_2(a) \end{pmatrix}$$

where  $\pi_1, \pi_2$  are the representations on  $H$  and  $K$ , respectively, and  $\delta : \mathcal{A} \rightarrow \mathcal{L}(K, H)$  is a derivation. A derivation  $\delta$  is called *inner* if there is an operator  $X$  in  $\mathcal{L}(K, H)$  such that  $\delta(a) = \pi_1(a)X - X\pi_2(a)$ ,  $a \in \mathcal{A}$ . It is easy to see that the derivation  $\delta$  is inner if and only if the sequence above is equivalent to the trivial sequence (i.e., the sequence where the module action on the direct sum is  $\pi_1 \oplus \pi_2$ ). By identifying the representation with the derivation one gets the usual Hochschild characterization of  $\text{Ext}_{\mathcal{A}}(H, K)$  as derivations modulo inner ones.

### 3. EXT OVER THE DISK ALGEBRA

Recall that an operator  $T$  on a Hilbert space  $H$  is polynomially bounded if and only if  $p \mapsto p(T)$  extends to a representation of the disk algebra,  $\mathbf{A}(\mathbf{D})$ , on  $H$ . On the other hand, given a representation  $\pi : \mathbf{A}(\mathbf{D}) \rightarrow \mathcal{L}(H)$ , the operator  $T = \pi(z)$  is polynomially bounded, where  $z$  is the function  $z \mapsto z$ . Note that  $\pi(p) = p(T)$  for all polynomials  $p$ . Because of this correspondence we will write  $(H, T)$  for the Hilbert module  $H$  with multiplication by  $z$  determined by the operator  $T$ .

Let  $(H, T_0)$  and  $(K, T_1)$  be two Hilbert  $\mathbf{A}(\mathbf{D})$ -modules. A derivation  $\delta : \mathbf{A}(\mathbf{D}) \rightarrow \mathcal{L}(K, H)$  is uniquely determined by the operator  $X = \delta(z)$  which, in turn, uniquely determines multiplication by  $z$  on  $H \oplus K$ . It is not hard to see that  $\delta$  is inner exactly when there is a  $Y \in \mathcal{L}(K, H)$  such that  $X = T_0Y - YT_1$ .

Let  $\text{PB}(K, H)$  denote the set of all  $X \in \mathcal{L}(K, H)$  such that the  $2 \times 2$  operator matrix

$$\begin{pmatrix} T_0 & X \\ 0 & T_1 \end{pmatrix}$$

is bounded on  $\mathcal{L}(H \oplus K)$ , and let  $\Delta(K, H)$  be the set of all commutators  $T_0Y - YT_1$  as  $Y$  ranges over  $\mathcal{L}(K, H)$ . It follows that  $\text{Ext}_{\mathbf{A}(D)}(K, H^2)$  is isomorphic to the quotient  $\text{PB}(K, H) / \Delta(K, H)$ .

### 4. $\text{Ext}_{\mathbf{A}(D)}(K, H^2)$

The Hardy space,  $H^2$ , is the Hilbert space of analytic functions on the disk satisfying

$$\|f\|^2 \equiv \sup_{0 < r < 1} \frac{1}{2\pi} \int_0^{2\pi} |f(re^{i\theta})|^2 d\theta < \infty.$$

We will frequently identify  $f \in H^2$  with its boundary values on the circle  $\mathbf{T}$ .  $P_+$  will denote the orthogonal projection of  $L^2(\mathbf{T})$  onto  $H^2$  and  $S$  will denote the forward shift on  $H^2$ . That is,  $Sf(z) = zf(z)$ ,  $f \in H^2$ . The operator  $S$  is contractive on  $H^2$  so, by von Neumann's inequality,  $(H^2, S)$  is a Hilbert  $\mathbf{A}(\mathbf{D})$ -module and the action is just pointwise multiplication.

Recall, for  $k \in K$  the rank one operator  $1 \otimes k \in \mathcal{L}(K, H^2)$  is defined by  $(1 \otimes k)f = \langle f, k \rangle_K e_0$ , where  $e_0$  is the constant function 1.

The proof of the following theorem appears in [2] and allows one to calculate the groups  $\text{Ext}_{A(D)}(K, H^2)$  for a large class of Hilbert modules  $K$ . A special case of the theorem appeared in [5].

**Theorem 1.** *Let  $T$  be a polynomially bounded operator on  $K$ . For  $X$  in  $\mathcal{L}(K, H^2)$ , let*

$$R(X) = \begin{pmatrix} S & X \\ 0 & T \end{pmatrix}.$$

The following are equivalent:

- 1)  $R(X)$  is power bounded on  $H^2 \oplus K$ .
- 2)  $R(X)$  is polynomially bounded on  $H^2 \oplus K$ .
- 3)  $\exists k \in K$  and  $Y \in \mathcal{L}(K, H^2)$  such that
  - i)  $X = 1 \otimes k + SY - YT$ , and
  - ii) For all  $f \in K$ ,  $\sum_{n=0}^{\infty} |\langle T^n f, k \rangle|^2 < \infty$ .

*Remarks.* (a) The first condition in 3) says that  $R(X)$  is similar to  $R(1 \otimes k)$  via the similarity  $\begin{pmatrix} I & Y \\ 0 & I \end{pmatrix}$ . Condition ii) is precisely the condition that  $R(1 \otimes k)$  is power bounded.

(b) This theorem tells us that the equivalence class of the operator  $X$  in  $\text{Ext}_{A(D)}(K, H^2)$  is determined by a rank one operator of the form  $1 \otimes k$  for some  $k \in K$ .

(c) The second condition in 3) is equivalent to the following: There exists  $W \in \mathcal{L}(H^2, K)$  such that  $WS = T^*W$  and  $W(1) = k$ . To see this note that if  $W$  intertwines  $S$  and  $T^*$  and sends 1 to  $k$ , then  $W(e_n) = T^{*n}k$ . So for  $f \in K$ ,  $\|W^*f\|^2 = \sum_{n=0}^{\infty} |\langle f, W e_n \rangle_K|^2 = \sum_{n=0}^{\infty} |\langle T^n f, k \rangle_K|^2$ .

(d) By remark (c), if we replace  $T$  by  $T^*$  in the theorem, then  $R(1 \otimes k)$  is polynomially bounded if and only if  $p \mapsto p(T)k$  extends to a bounded operator from  $H^2$  into  $K$ . So if  $K$  is a functional Hilbert space such that  $Tf(z) \equiv zf(z)$  is polynomially bounded, then the theorem gives us an alternative criterion for a function  $k \in K$  to be an analytic multiplier of  $H^2$  into  $K$ .

Recall that an analytic reproducing kernel Hilbert space on the disk is a Hilbert space  $H$  of analytic functions on the disk such that for each  $|w| < 1$  the functional  $f \mapsto f(w)$  is bounded on  $H$ . By the Riesz Representation Theorem there exist functions  $k_w \in H$  such that  $f(w) = \langle f, k_w \rangle_H$  for all  $w$  in the disk. The function  $K(z, w) = k_w(z)$  is called the kernel function for  $H$  and we will write  $H(K)$  instead of  $H$  since the kernel uniquely determines the Hilbert space  $H$ .

**Corollary 1.** *Let  $H(K)$  be an analytic reproducing kernel Hilbert space on the disk such that  $Tf(z) \equiv zf(z)$  is polynomially bounded on  $H(K)$ . Let  $\overline{H(K)}$  denote the Hilbert module  $(H(K), T^*)$ . Then  $\text{Ext}_{A(D)}(\overline{H(K)}, H^2)$  can be identified with  $\mathcal{M}(H^2, H(K))$ , the set of pointwise multipliers from  $H^2$  into  $H(K)$ .*

*Proof.* Suppose  $1 \otimes g = SY - YT^*$  for some bounded  $Y$ . Multiplying on the left by  $S^*$  yields  $Y = S^*YT^*$ . So  $\forall |w| < 1$ ,  $0 = (Y - S^*YT^*)k_w = (1 - \overline{w}S^*)Yk_w$ . Therefore,  $Yk_w = 0 \ \forall |w| < 1$ . Since the kernel functions  $k_w$  span  $H(K)$ ,  $Y = 0$ . It follows from remark (c) above that  $\text{Ext}_{A(D)}(\overline{H(K)}, H^2)$  can be identified with the space of functions  $g \in H^2(\beta)$  such that  $p \mapsto p(T)g$  extends to a bounded operator

from  $H^2$  into  $H^2(\beta)$ . It is easy to show, using the Closed Graph Theorem together with the fact that the evaluation functionals are continuous on both  $H^2$  and  $H(K)$ , that  $g \in \mathcal{M}(H^2, H^2(\beta))$ .  $\square$

Let  $\{\beta_n\}$  be a sequence of positive numbers with  $\beta_0 = 1$  and such that  $\sup_{n \geq 0} \beta_n/\beta_{n+1} < \infty$ . Then  $H^2(\beta)$  is defined to be the Hilbert space of analytic functions  $f(z) = \sum_{n=0}^\infty a_n z^n$  such that

$$\|f\|^2 \equiv \sum_{n=0}^\infty \frac{|a_n|^2}{\beta_n} < \infty.$$

It is well known, [7], that  $Tf(z) = zf(z)$  is bounded on  $H^2(\beta)$  and unitarily equivalent to the weighted shift on  $H^2$  with weight sequence  $\{\sqrt{\beta_n/\beta_{n+1}}\}$ . Throughout, we will assume that  $T$  is a contraction so that  $(H^2(\beta), T)$  is a Hilbert  $\mathbf{A}(\mathbf{D})$ -module.

We will use the following notation in the proposition below. For  $g \in H^2$ ,  $\Gamma_g$  will denote the Hankel matrix with symbol  $g$ . That is

$$\Gamma_g = (\hat{g}(i + j))_{i,j \geq 0},$$

where  $\hat{g}(n)$  are the Fourier coefficients of  $g$ .

For  $\varphi \in L^\infty(T)$ , let  $T_\varphi$  denote the Toeplitz matrix

$$T_\varphi = (\hat{\varphi}(i - j))_{i,j \geq 0}.$$

Finally,  $D_\beta$  will denote the diagonal matrix,  $diag(\sqrt{\beta_n})_{n \geq 0}$ .

**Proposition 1.** *For  $g$  in  $H^2(\beta)$ , define  $g_1$  in  $H^2$  by  $\hat{g}_1(n) = \hat{g}(n)/\beta_n$ . Let*

$$R(1 \otimes g) = \begin{pmatrix} S & 1 \otimes g \\ 0 & T \end{pmatrix}.$$

(1)  *$R(1 \otimes g)$  is polynomially bounded on  $H^2 \oplus H^2(\beta)$  if and only if the weighted Hankel matrix  $D_\beta \Gamma_{g_1}$  is bounded on  $\ell^2$ .*

(2) *There is an operator  $Y$  in  $\mathcal{L}(H^2(\beta), H^2)$  such that  $1 \otimes g = SY - YT$  if and only if  $\exists \varphi \in L^\infty(\mathbf{T})$  satisfying*

- a)  $P_+(e^{i\theta} \varphi(e^{i\theta})) = -g_1$ , and
- b) *The weighted Toeplitz matrix  $T_\varphi D_\beta$  is bounded on  $\ell^2$ .*

*Proof.* By remark (c), the matrix  $R(1 \otimes g)$  is polynomially bounded if and only if the operator  $We_j = T^{*j}g$  extends to a bounded operator from  $H^2$  into  $H^2(\beta)$ . One checks that the matrix for this operator with respect to the usual orthonormal bases,  $\{e_n\}_{n \geq 0}$  for  $H^2$  and  $\{\sqrt{\beta_n}z^n\}_{n \geq 0}$  for  $H^2(\beta)$ , is the weighted Hankel matrix above. For the proof of (2), suppose  $1 \otimes g = SY - YT$  for some  $Y \in \mathcal{L}(H^2(\beta), H^2)$ . Multiplying on the left by  $S^*$ , we see that  $Y = S^*YT$ . It follows that the matrix  $(\langle Yz^j, e_i \rangle)_{i,j \geq 0}$  is Toeplitz. To see that the matrix is bounded, note that  $T$  is a contraction so that  $D_\beta^{-1}$  is bounded. Hence  $D_\beta^{-1}Y$  is bounded. But this is the Toeplitz matrix above. From the commutator equation one has  $\hat{g}_1(n) = -\langle Y^*e_0, z^{n+1} \rangle_\beta$ .  $\square$

**Corollary 2.**  $\text{Ext}_{A(D)}(H^2, H^2) = (0)$ .

*Proof.* By setting  $\beta_n = 1$ , one sees that  $R(1 \otimes g)$  is polynomially bounded if and only if the Hankel matrix  $\Gamma_g$  is bounded on  $\ell^2$ . By Nehari's theorem, this happens if and only if  $g = P_+\psi$  for some  $\psi \in L^\infty(\mathbf{T})$ .  $\square$

**Corollary 3.** *If  $\beta_n \rightarrow +\infty$ , then we may identify  $\text{Ext}_{A(D)}(H^2(\beta), H^2)$  as the vector space of functions  $g \in H^2(\beta)$  such that the matrix  $D_\beta \Gamma_{g_1}$  is bounded on  $\ell^2$ .*

*Proof.* Note that the diagonal matrix  $D_\beta^{-1} = \text{diag}(\beta_j^{-1/2})_{j \geq 0}$  is compact. So if  $T_\varphi D_\beta$  is bounded, then  $T_\varphi$  is compact and this implies  $\varphi = 0$ . Therefore, no nonzero rank one operator of the form  $1 \otimes g$  is a commutator.  $\square$

*Remark.* If we take  $\beta_n = n + 1$ , then  $H^2(\beta)$  is the Bergman space,  $L_a^2$ , and  $g' \in \text{Ext}_{A(D)}(L_a^2, H^2)$  if and only if the matrix  $\text{diag}(\sqrt{n+1})_{n \geq 0} \Gamma_g$  is bounded on  $\ell^2$ . This happens if and only if the range of  $\Gamma_g$  is contained in the Dirichlet space  $= H^2(\frac{1}{\beta})$ . It is not hard to see that  $\text{Ext}_{A(D)}(L_a^2, H^2)$  contains all Bloch functions. In fact, if  $T$  is pointwise multiplication by  $z$  on  $L_a^2$ , then the operator  $W$  defined on polynomials by  $W(p) = p(T^*)g$  is just the restriction of the little Hankel (with symbol  $g$ ) on the Bergman space to  $H^2$ . It is well known that this operator is bounded on  $L_a^2$  if and only if  $g$  is Bloch, see [8]. It will follow from a result in the next section that  $\mathcal{M}(H^2, L_a^2)$  is also contained in  $\text{Ext}_{A(D)}(L_a^2, H^2)$ .

5. A PROOF OF BOURGAIN’S RESULT

Bourgain in [1] proved that the operator

$$R_f = \begin{pmatrix} S^* & \Gamma_f \\ 0 & S \end{pmatrix}$$

is similar to a contraction if  $f' \in \text{BMOA}$ . In this section we will give an alternative proof of this result using subnormality and a characterization of BMOA in terms of Carleson measures on the Hardy space. What we will prove is the following:

(2)  $f'$  is BMOA  $\implies$  the matrix  $A = ((i+1)\hat{f}(i+j))_{i,j \geq 0}$  is bounded on  $\ell^2$ .

Note that if  $A$  is bounded, then  $\Gamma_f = A - SAS = S^*(SA) - (SA)S$ . Hence,  $R_f$  is similar to  $S^* \oplus S$  via the similarity

$$\begin{pmatrix} I & SA \\ 0 & I \end{pmatrix}.$$

*Proof of (2).*  $\square$

**Lemma 1.** *Suppose that  $T$  is subnormal on a Hilbert space  $K$  and let  $f \in K$ . If  $W(p) \equiv p(T)f$  extends to a bounded operator from  $H^2$  into  $K$ , then the operator  $V(p) \equiv p(T^*)f$  extends to a bounded operator from  $H^2$  into  $K$ .*

*Proof.* The transpose of the matrix for  $W^*W$  is

$$B = (\langle T^{*j}T^i f, f \rangle_K)_{i,j \geq 0}$$

and the the matrix for  $V^*V$  is

$$C = (\langle T^i T^{*j} f, f \rangle_K)_{i,j \geq 0}.$$

By the Bram-Halmos criterion for subnormality,  $B \geq C$ .  $\square$

*Remark.* If  $Tf(z) \equiv zf(z)$  is polynomially bounded on  $H^2(\beta)$  and  $\beta_n \rightarrow \infty$ , then by Corollary 3.2,  $\text{Ext}_{A(D)}(H^2(\beta), H^2)$  is the vector space of functions  $g \in H^2(\beta)$  such that  $p \mapsto p(T^*)g$  extends to a bounded operator from  $H^2$  into  $H^2(\beta)$ . If we suppose further that  $T$  is subnormal, then it follows from Lemma 4.1 that  $\mathcal{M}(H^2, H^2(\beta))$  is contained in  $\text{Ext}_{A(D)}(H^2(\beta), H^2)$ .

By Nehari's theorem a function  $f \in H^2$  is BMOA if and only if the Hankel matrix  $\Gamma_f$  is bounded on  $\ell^2$ . Another useful criterion for  $f$  to be BMOA is that the measure  $|f'(z)|^2 \log \frac{1}{|z|} dA(z)$  is Carleson for  $H^2$ , where  $A$  is normalized area measure on the disk. A good reference for this is Zhu's book [8]. Now what this means is that  $f'$  is a pointwise multiplier from  $H^2$  into  $\mathcal{P}^2(\mu)$ , the closure of the analytic polynomials in  $L^2(\mu)$ , where  $d\mu(z) = \log \frac{1}{|z|} dA(z)$ . One verifies that  $\frac{1}{(n+1)^2} = \int_D |z|^{2n} \log \frac{1}{|z|} dA(z)$  so that  $\mathcal{P}^2(\mu) = H^2(\beta)$  with  $\beta_n = (n+1)^2$ , and multiplication by  $z$  is subnormal on this space. By the remark above  $f' \in \text{Ext}_{A(D)}(H^2(\beta), H^2)$ . So by Proposition 4.1, the matrix

$$\left( \frac{i+1}{i+j+1} \hat{f}(i+j) \right)_{i,j \geq 0}$$

must be bounded on  $\ell^2$ . If we repeat this argument with  $f'$  in place of  $f$ , we get that the matrix  $A$  is bounded.  $\square$

#### ACKNOWLEDGEMENTS

I would like to thank my advisor, Vern Paulsen, for his guidance and advice while working on this project.

#### REFERENCES

- [1] J. Bourgain, *The Similarity Problem for polynomially bounded operators on Hilbert space*, Israel Journal of Math., Vol. 54, No. 2, (1986). MR **88h**:47024
- [2] J. F. Carlson and D. N. Clark, *Cohomology and extensions of Hilbert modules*, J. Functional Analysis 128 (1995).
- [3] J. F. Carlson, D. N. Clark, C. Foias and J. P. Williams, *Projective Hilbert  $A(D)$ -Modules*, New York J. Math. 1 (1994), 26–38. CMP 94:15
- [4] R. G. Douglas and V. I. Paulsen, *Hilbert modules over function algebras*, Longman Scientific & Technical, New York (1989). MR **91g**:46084
- [5] C. Foias and J. P. Williams, *On a class of polynomially bounded operators*, preprint (1974).
- [6] V. V. Peller, *Estimates of functions of power bounded operators on Hilbert spaces*, J. Operator Theory, 7 (1982), 341–372. MR **83i**:47019
- [7] A. Shields, *Weighted shift operators and analytic function theory*, Topics in Operator Theory, Math. Surveys, No. 13, Amer. Math. Soc., Providence, R.I. (1974) MR **50**:14341
- [8] K. Zhu, *Operator Theory in function spaces*, Marcel Dekker, Inc., New York and Basel (1990). MR **92c**:47031

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF HOUSTON, HOUSTON, TEXAS 77204-3476  
*E-mail address:* sarah@math.uh.edu