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MEAN THEORETIC APPROACH TO THE GRAND FURUTA INEQUALITY

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Dedicated to Professor Tsuyoshi Ando, the originator of the theory of operator means, on his retirement from Hokkaido University

ABSTRACT. Very recently, Furuta obtained the grand Furuta inequality which is a parameteric formula interpolating the Furuta inequality and the Ando-Hiai inequality as follows : If $A \geq B \geq 0$ and A is invertible, then for each $t \in [0,1]$,

$$F_{p,t}(A,B,r,s) = A^{-r/2}\{A^{r/2}(A^{-t/2}B^pA^{-t/2})^sA^{r/2}\}^{\frac{1-t+r}{(p-t)s+r}}A^{-r/2}$$

is a decreasing function of both r and s for all $r \geq t,\ p \geq 1$ and $s \geq 1$. In this note, we employ a mean theoretic approach to the grand Furuta inequality. Consequently we propose a basic inequality, by which we present a simple proof of the grand Furuta inequality.

1. INTRODUCTION

Throughout this note, we consider bounded linear operators acting on a Hilbert space, simply operators. An operator A is positive if $(Ax,x) \geq 0$ for all $x \in H$. The Löwner-Heinz inequality says that the function $t \to t^\alpha$ on $[0,\infty)$ is operator monotone for $0 \leq \alpha \leq 1$, i.e.,

$$A \geq B \geq 0 \quad \text{implies} \quad A^\alpha \geq B^\alpha \tag{1}$$

(cf. [13] and [15]). Furuta [7] gave it an ingenious extension which is called the Furuta inequality : If $A \geq B \geq 0$, then

$$(A^rA^pA^r)^{1/q} \geq (A^rB^pA^r)^{1/q} \tag{2}$$

holds for $r \geq 0,\ p \geq 0$ and $q \geq 1$ with $(1+2r)q \geq p+2r$; see [8] for an elementary proof.

Recently Ando and Hiai [2] discussed the log-majorization for positive operators and obtained the following fundamental inequality [2, Theorem 3.5], which is equivalent to their main log-majorization theorem [2, Theorem 2.1]. If $A \geq B \geq 0$ and A is invertible, then the following holds for $p,r \geq 1$:

$$A^r \geq \{A^{r/2}(A^{-1/2}B^pA^{-1/2})^rA^{r/2}\}^{1/p}. \tag{3}$$

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Very recently, Furuta [12] obtained a parameteric formula interpolating the Furuta inequality (2) and the Ando-Hiai inequality (3) in the following manner:

The grand Furuta inequality ([12]). *If $A \geq B \geq 0$ and A is invertible, then for each $t \in [0,1]$,*

$$F_{p,t}(A,B,r,s) = A^{-r/2}\{A^{r/2}(A^{-t/2}B^pA^{-t/2})^sA^{r/2}\}^{\frac{1-t+r}{(p-t)s+r}}A^{-r/2} \tag{4}$$

is a decreasing function of both r and s for all $r \geq t$, $p \geq 1$ and $s \geq 1$.

In particular, the inequality

$$A^{1-t} = F_{p,t}(A,A,r,s) \geq F_{p,t}(A,B,r,s) \tag{5}$$

holds for $r \geq t$, $p \geq 1$ and $s \geq 1$.

As a matter of fact, (2) and (3) appear in the grand Furuta inequality as the extremal cases $t = 0$ and $t = 1$ with $r = s$ respectively. Therefore we call it the grand Furuta inequality. We note that the original proof in [12] is quite elementary but somewhat technical.

In this note, we employ a mean theoretic approach to the grand Furuta inequality as has been done for the Furuta inequality (see [3], [4], [5], [6], [9], [14], [15]). Thus we propose a basic inequality, by which we present a simple proof of the grand Furuta inequality.

2. Means of operators

The theory of operator means was established by Kubo and Ando [16], whose heart is the correspondence between operator monotone functions f and means m given by

$$f(t) = 1 \ m \ t \quad (t > 0).$$

In connection with the Löwner-Heinz inequality (1), they exhibit means $\sharp_s$ for $0 \leq s \leq 1$ such that $1 \sharp_s t = t^s$ $(t > 0)$, more precisely

$$A \sharp_s B = A^{1/2}(A^{-1/2}BA^{-1/2})^sA^{1/2} \tag{6}$$

for positive invertible operators A and B.

In [10], Furuta proved the monotonicity of the function

$$F(p) = (B^rA^pB^r)^{\frac{1+2r}{p+2r}} \quad \text{for} \quad p \geq 1 \tag{7}$$

associated with his inequality under the assumption $A \geq B \geq 0$ and $r \geq 0$. Noting that the monotonicity of $F(p)$ can be rephrased in terms of the function

$$M(p,r) = B^{-2r} \sharp_{\frac{1+2r}{p+2r}} A^p, \tag{8}$$

we showed that $M(p,r)$ is an increasing function of both p and r for all $p \geq 1$ and $r \geq 0$ [4, Theorem 1]; see Lemma 5 below. Moreover, we pointed out that its modification includes Ando's theorem on the geometric mean in [1].

We now rewrite (4) in a mean theoretic form, using the same technique that we used to rewrite (7) as (8). For the sake of convenience, we define $\natural_s$ for $s \in \mathbb{R}$ as in [12] as an extension of $\sharp_s$ for $0 \leq s \leq 1$,

$$A \natural_s B = A^{1/2}(A^{-1/2}BA^{-1/2})^sA^{1/2} \tag{6'}$$

for positive invertible operators A and B. For given $A \geq B \geq 0,\ p \geq 1$ and $t \in [0,1]$, we let

$$F(r,s) = A^{-r+t} \ \sharp_{\frac{1-t+r}{(p-t)s+r}} \ (A^t \ \natural_s \ B^p) \tag{9}$$

for $r \geq t$ and $s \geq 1$. It is easily checked that $F(r,s) = A^{t/2} F_{p,t}(A,B,r,s) A^{t/2}$ and so the monotonicity of $F_{p,t}(A,B,r,s)$ in (4) is equivalent to that of $F(r,s)$. Similarly, (5) can be rewritten in the form

$$F(r,s) \leq A \quad \text{for} \quad r \geq t \quad \text{and} \quad s \geq 1 \tag{10}$$

if $A \geq B \geq 0$ and A is invertible.

Concluding this section, we note that $\natural_s$ is multiplicative in the sense that

$$A \ \natural_{ab} \ B = A \ \natural_a \ (A \ \natural_b \ B) \tag{11}$$

for all a and b.

3. The grand Furuta inequality

We begin by stating two simple lemmas by Furuta in [10] and [12]. For the sake of convenience, we give them short proofs.

Lemma 0 ([10]). *For positive operators A and B,*

$$(ABA)^s = AB^{1/2}(B^{1/2}A^2B^{1/2})^{s-1}B^{1/2}A$$

holds for $s \geq 1$.

Proof. Let $AB^{\frac{1}{2}} = UH$ be the polar decomposition of $AB^{\frac{1}{2}}$. Then we have for $s \geq 0$

$$\begin{aligned}(ABA)^{1+s} &= (UH^2U^*)^{1+s} = UH^{2+2s}U^* \\ &= UHH^{2s}HU^* = AB^{1/2}(B^{1/2}A^2B^{1/2})^s B^{1/2}A.\end{aligned}$$

Lemma 1 ([12]). *If $A \geq B \geq 0$ and A is invertible, then*

$$A^t \ \natural_s \ B^p \leq B^{(p-t)s+t}$$

for $p \geq 1,\ 1 \leq s \leq 2$ and $0 \leq t \leq 1$.

Proof. It follows from (1) that $A^{-t} \leq B^{-t}$ and moreover

$$\begin{aligned}A^t \ \natural_s \ B^p &= A^{t/2}(A^{-t/2}B^pA^{-t/2})^s A^{t/2} \\ &= A^{t/2}A^{-t/2}B^{p/2}(B^{p/2}A^{-t}B^{p/2})^{s-1}B^{p/2}A^{-t/2}A^{t/2} \text{ by Lemma 0} \\ &\leq B^{p/2}(B^{p/2}B^{-t}B^{p/2})^{s-1}B^{p/2} \\ &= B^{(p-t)s+t}.\end{aligned}$$

We use Lemma 1 to prove the following basic inequality which will work well in a proof of the grand Furuta inequality.

Theorem 2. *If $A \geq B \geq 0$ and A is invertible, then*

$$(A^t \ \natural_s \ B^p)^{1/(s(p-t)+t)} \leq B$$

for $p \geq 1$, $s \geq 1$ and $0 \leq t \leq 1$.

Proof. It suffices to show that

$$(A^t \natural_{2^k s} B^p)^{1/(2^k s(p-t)+t)} \le B \tag{12}$$

for $1 \le s \le 2$ and $k = 1, 2, \cdots$. Lemma 1 says that (12) holds for $k = 0$. So we put $p_1 = s(p-t)+t$, $B_1 = (A^t \natural_s B^p)^{1/p_1}$ and inductively

$$p_{k+1} = 2^k s(p-t)+t \quad \text{and} \quad B_{k+1} = (A^t \natural_{2^k s} B^p)^{1/p_{k+1}}$$

for $k = 1, 2, \cdots$. Then we have $p_{k+1} = 2(p_k - t) + t$ and by (11)

$$B_{k+1} = (A^t \natural_2 (A^t \natural_{2^{k-1}s} B^p))^{1/p_{k+1}} = (A^t \natural_2 B_k^{p_k})^{1/p_{k+1}}$$

for $k = 1, 2, \cdots$.

It therefore suffices to prove that $B_{k+1} \le B_k$. For a fixed k, we can assume that $B_k \le A$ because $B_1 \le B \le A$. Hence we apply Lemma 1 to $B_k \le A$, $p = p_k$ and $s = 2$. It implies that

$$(A^t \natural_2 B_k^{p_k})^{1/(2(p_k-t)+t)} \le B_k.$$

Finally, since the left-hand side of this is just B_{k+1} by the above remark, we obtain the conclusion $B_{k+1} \le B_k$.

The second tool is a special case in the Furuta inequality (2).

Lemma 3. *If $A \ge B \ge 0$, then*

$$A \ge (A^{b/2} B^p A^{b/2})^{1/(p+b)}$$

for $p \ge 1$ and $b \ge 0$.

Proof. We take $r = b/2$ and $q = p + b$ in (2). Since p, q and r satisfy the condition which ensures (2), we have the desired inequality.

Based on Theorem 2 and Lemma 3, we give a simple proof of the statement that $F(r, s)$ is a decreasing function of r. The proof we give is similar to that of [4, Theorem 1].

Lemma 4. *If $A \ge B \ge 0$, then $F(r, s)$ is a decreasing function of r for all $r \ge t$.*

Proof. First of all, we put $B_1 = (A^t \natural_s B^p)^{1/((p-t)s+t)}$ as in Theorem 2 and $\langle s \rangle = \frac{1-t+r}{(p-t)s+r}$ for given $s \ge 1$, p, r and t. Since $B_1 \le B \le A$ by Theorem 2, Lemma 3 implies that

$$D = (A^{(r-t+d)/2} B_1^{(p-t)s+t} A^{(r-t+d)/2})^{1/((p-t)s+r+d)} \le A,$$

so that $D^d \le A^d$ by (1) for $0 < d < 1$. Therefore we have

$$\begin{aligned}
F(r, s) &= A^{-r+t} \sharp_{\langle s \rangle} B_1^{(p-t)s+t} \\
&= A^{-(r+d-t)/2}(A^d \sharp_{\langle s \rangle} A^{(r+d-t)/2} B_1^{(p-t)s+t} A^{(r+d-t)/2}) A^{-(r+d-t)/2} \\
&\ge A^{-(r+d-t)/2}(D^d \sharp_{\langle s \rangle} A^{(r+d-t)/2} B_1^{(p-t)s+t} A^{(r+d-t)/2}) A^{-(r+d-t)/2} \\
&= A^{-(r+d-t)/2}(D^d \sharp_{\langle s \rangle} D^{(p-t)s+r+d}) A^{-(r+d-t)/2} \\
&= A^{-(r+d-t)/2} D^{1-t+r+d} A^{-(r+d-t)/2} \\
&= A^{-(r+d)+t} \sharp_{\frac{1-t+r+d}{(p-t)s+r+d}} B_1^{(p-t)s+t} \\
&= F(r+d, s),
\end{aligned}$$

so the proof is complete.

To prove that $F(r,s)$ is a decreasing function of s, we need the following result equivalent to [4, Theorem 3]; cf. [11].

Lemma 5. *If $A \geq B \geq 0$ and $\gamma \geq 0$ is given, then the operator function*

$$f(\alpha,\beta) = A^{-\alpha} \sharp_{\frac{\gamma+\alpha}{\beta+\alpha}} B^{\beta}$$

is a decreasing function of both α and β for all $\alpha \geq 0$ and $\beta \geq \gamma$.

Proof. In [4, Theorem 3], we showed that if $C \geq D \geq 0$ and $\gamma \geq 0$ is given, then

$$g(\alpha,\beta) = D^{-\alpha} \sharp_{\frac{\gamma+\alpha}{\beta+\alpha}} C^{\beta}$$

is an increasing function of both α and β for all $\alpha \geq 0$ and $\beta \geq \gamma$. Since $B^{-1} \geq A^{-1} \geq 0$, it implies that

$$h(\alpha,\beta) = (A^{-1})^{-\alpha} \sharp_{\frac{\gamma+\alpha}{\beta+\alpha}} (B^{-1})^{\beta} = A^{\alpha} \sharp_{\frac{\gamma+\alpha}{\beta+\alpha}} B^{-\beta}$$

is also an increasing function of both α and β for all $\alpha \geq 0$ and $\beta \geq \gamma$. Taking the inverse, we obtain that

$$f(\alpha,\beta) = A^{-\alpha} \sharp_{\frac{\gamma+\alpha}{\beta+\alpha}} B^{\beta} = (A^{\alpha} \sharp_{\frac{\gamma+\alpha}{\beta+\alpha}} B^{-\beta})^{-1} = (h(\alpha,\beta))^{-1}$$

is a decreasing function of both α and β for all $\alpha \geq 0$ and $\beta \geq \gamma$, as required.

Lemma 6. *If $A \geq B \geq 0$, then $F(r,s)$ is a decreasing function of s for all $s \geq 1$.*

Proof. Since $B_1 \leq A$ by Theorem 2, we can apply Lemma 1 to B_1 and A. Namely we have, for $1 \leq s_1 \leq 2$

$$A^t \natural_{s_1} B_1^{p_1} \leq B_1^{(p_1-t)s_1+t}$$

for $p_1 \geq 1$. Taking $p_1 = (p-t)s+t \geq 1$ in particular, we have

$$A^t \natural_{s_1} B_1^{(p-t)s+t} \leq B_1^{(p-t)ss_1+t}.$$

On the other hand, since the left-hand side above is of the form

$$A^t \natural_{s_1} B_1^{(p-t)s+t} = A^t \natural_{s_1} (A^t \natural_s B^p) = A^t \natural_{ss_1} B^p,$$

it follows that

$$A^t \natural_{ss_1} B^p \leq B_1^{(p-t)ss_1+t}.$$

Hence the monotonicity of operator means implies that

$$A^{-r+t} \sharp_{\frac{(p-t)s+r}{(p-t)ss_1+r}} (A^t \natural_{ss_1} B^p) \leq A^{-r+t} \sharp_{\frac{(p-t)s+r}{(p-t)ss_1+r}} B_1^{(p-t)ss_1+t}. \tag{13}$$

Now we apply Lemma 5 to $A \geq B_1$ for $\alpha = r-t, \beta = (p-t)ss_1+t$ and $\gamma = (p-t)s+t$. Then $\gamma \leq \beta$ by $1 \leq s_1$. That is, we have

$$A^{-r+t} \sharp_{\frac{(p-t)s+r}{(p-t)ss_1+r}} B_1^{(p-t)ss_1+t} \leq B_1^{(p-t)s+t} = A^t \natural_s B^p. \tag{14}$$

Combining (13) with (14), we obtain that

$$A^{-r+t} \sharp_{\frac{(p-t)s+r}{(p-t)ss_1+r}} (A^t \natural_{ss_1} B^p) \leq A^t \natural_s B^p.$$

Finally it implies that

$$\begin{aligned} F(r, ss_1) &= A^{-r+t} \sharp_{\frac{1-t+r}{(p-t)ss_1+r}} (A^t \natural_{ss_1} B^p) \\ &= A^{-r+t} \sharp_{\frac{1-t+r}{(p-t)s+r}} (A^{-r+t} \sharp_{\frac{(p-t)s+r}{(p-t)ss_1+r}} (A^t \natural_{ss_1} B^p)) \\ &\le A^{-r+t} \sharp_{\frac{1-t+r}{(p-t)s+r}} (A^t \natural_s B^p) \\ &= F(r, s), \end{aligned}$$

so the proof is complete.

Consequently the grand Furuta inequality is proved by Lemmas 4 and 6, in the form of (9).

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