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REMARKS ON THE LOCAL HOPF'S LEMMA

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Abstract. The paper deals with the problem of extending the recent work of M.S.Baouendi and L.P.Rothschild concerning harmonic functions vanishing to infinite order in the normal direction in balls and half-spaces. Contrary to what one expects, we show that the B.-R. result extends neither to arbitrary domains nor to cases when the normal is replaced by a curve transversal to the boundary. The exact criterion when the result holds in $\mathbf{R}^2$ is given.

0. Introduction

Consider a domain D in $\mathbf{R}^n$ with a smooth boundary Γ. Let Ω be an open neighborhood of a point x_0 on Γ. Set $\Omega^+ = \Omega \cap D$, $\Omega^- = \Omega \setminus \bar{D}$, $V = \Omega \cap \Gamma$. Assume that Ω^+, Ω^- are connected and V is real analytic.

A continuous function u in $\overline{\Omega^+}$ is said to vanish to infinite order at x_0 if for every positive integer N:

$$\lim_{\substack{x \in \Omega^+ \\ x \to x_0}} \frac{u(x)}{|x - x_0|^N} = 0. \tag{0.1}$$

Also, u vanishes to infinite order in a direction $\vec{b}$ at x_0, where $\vec{b}$ is a unit vector pointing inside Ω^+ and transversal to Γ, if for every N:

$$\lim_{t \to 0^+} \frac{u(x_0 + t\vec{b})}{t^N} = 0. \tag{0.2}$$

Finally, u vanishes to infinite order on a non-singular smooth arc S: $x = \gamma(t)$, $0 \le t \le 1$, in Ω^+ passing through x_0 and transversal to Γ if for all N:

$$\lim_{t \to 0^+} \frac{u(x(t))}{t^N} = 0. \tag{0.3}$$

(One can easily check that the definition (0.3) does not depend on the parametrization.)

M.S.Baouendi and L.P.Rothschild proved in [1] the following theorem.

(A). *Let D be either a ball or a half-space in $\mathbf{R}^n$. If u is harmonic in Ω^+ and continuous in $\overline{\Omega^+}$, vanishes to infinite order in the normal direction at x_0, and $u(x) \ge 0$ for $x \in V$, then $u(x) \equiv 0$ in some neighborhood of x_0 in V.*

This theorem implies the corollary

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(B). *If, in addition to the hypotheses in* (A)*, u vanishes to infinite order at x_0, then $u(x) \equiv 0$ in Ω^+.*

Earlier, somewhat similar but weaker results under the hypothesis that u is harmonic in a half-plane and decays exponentially along the y-axis were obtained in [4]. Also see further extensions by H.Shapiro [5], [6].

The authors in [1] remarked that similar results should, perhaps, hold in arbitrary domains D (in fact, only a small real-analytic piece V of the boundary Γ matters). An interesting question is also whether one can replace the normal direction by an oblique one, or even by a real-analytic or C^∞ curve transversal to Γ. More precisely, does the following generalization of (A) hold?

(A′). *If u is harmonic in Ω^+ and continuous in $\overline{\Omega^+}$, vanishes to infinite order in a direction $\vec{b}$ (or on an arc S) at x_0, and $u(x) \geq 0$ for $x \in V$, then $u(x) \equiv 0$ in some neighborhood of x_0 in V.*

Accordingly, a generalization of Corollary (B) is:

(B′). *If, in addition to the hypotheses in* (A′)*, u vanishes to infinite order at x_0, then $u(x) \equiv 0$ in Ω^+.*

It turns out that in general answers to the questions about (A′) are in the negative. For general domains and transversal curves (A′) does not hold. The subject of this paper is characterization of plane domains for which (A′) is still true. Our main result (Theorem 2) is that (A′) holds if and only if V is locally symmetric about the normal to Γ at x_0. However, (B′) holds for all domains. We also include some results for the situation when the normal is replaced by a transversal curve.

1. Examples of failure of (A′)

As usual we identify $\mathbf{R}^2$ with $\mathbf{C}$, $x + iy = z \in \mathbf{C}$, $x, y \in \mathbf{R}$. Let V contain $z = 0$ and be defined by

$$y = f(x), \tag{1.1}$$

where $f(x)$ is a real-analytic function, $f(0) = 0$ and the domain D near the origin lies "above" V. Consider the above problem with $u(x, y)$ vanishing to infinite order on the y-axis ($y \geq 0$) at $z = 0$. $f(x)$ has Taylor expansion near $x = 0$

$$f(x) = \sum_{k=1}^{\infty} c_k x^k, \tag{1.2}$$

where the coefficients c_k are real. If $c_1 \neq 0$, the y-axis is oblique with respect to V. If, in particular, all c_k, $k \geq 2$, vanish, V is a piece of the straight line $y = c_1 x$ and we have the half-plane situation with u vanishing to infinite order in an oblique direction. If $c_1 = f'(0) = 0$, i.e. V is tangent to the x-axis at $z = 0$, then the y-axis is normal to V. We can rewrite (1.2) as

$$f(x) = \sum_{k=k_0}^{\infty} c_k x^k, \tag{1.3}$$

where $k_0 \geq 1$, and $c_{k_0} \neq 0$ is the first non-vanishing coefficient in (1.2). The harmonic function

$$u(x, y) = \operatorname{Im} \sqrt{1 + z^2} \tag{1.4}$$

is continuous in a sufficiently small neighborhood of the origin (we choose the branch of $\sqrt{z}$ so that $\sqrt{1+0^2}=1$). On the y-axis $z=iy$, so $u(x,y)=\operatorname{Im}\sqrt{1-y^2}\equiv 0$ (for $|y|<1$) and therefore $u(x,y)$ vanishes to infinite order on the y-axis at $z=0$. To check the hypotheses of (A$'$) it remains to show that $u(x)$ has a constant sign on V.

Lemma 1.1. *If $k_0=2p+1$, where $p\geq 0$ is an integer and $c_{k_0}>0$ (or $c_{k_0}<0$), then $u(x,y)>0$ ($u(x,y)<0$) in a neighborhood of 0 for $z\neq 0$ on V.*

Proof. Without loss of generality suppose $c:=c_{k_0}>0$. From (1.3) we have:

$$f(x)=cx^{2p+1}+O(x^{2p+2}). \tag{1.5}$$

Let us use the notation $O_R(...)$ to say that $O(...)$ is real-valued. Then for $z\in V$ we have $z=x+i(cx^{2p+1}+O_R(x^{2p+2}))$ and

$$\begin{aligned}
u(x,y) &= \operatorname{Im}\sqrt{1+\{x+i[cx^{2p+1}+O_R(x^{2p+2})]\}^2}\\
&= \operatorname{Im}\sqrt{1+O_R(x^2)+2icx^{2p+2}+O(x^{2p+3})}\\
&= \operatorname{Im}(1+O_R(x^2)+icx^{2p+2}+O(x^{2p+3}))=cx^{2p+2}+O_R(x^{2p+3}),
\end{aligned}$$

so

$$u(x,f(x))=cx^{2p+2}+O_R(x^{2p+3}). \tag{1.6}$$

Thus, for sufficiently small $|x|\neq 0$, $\quad u(x,f(x))>0$. Therefore, all the hypotheses of (A$'$) are satisfied but (A$'$) nevertheless fails (u does not vanish anywhere on V near 0). Thus, we have the following corollaries:

Corollary 1.2. (A$'$) *fails if one replaces the normal by an arbitrary line oblique with respect to V.*

Corollary 1.3. (A$'$) *for an arbitrary domain D fails for all V as in* (1.5) *provided that $p\geq 1$ ($k_0\geq 3$).*

In general, $u(x,y)$ does not have to vanish not only on V but also on the y-axis (i.e., on a normal or oblique line) and, moreover, does not have to extend across V as a real-analytic function, contrary to what the Baouendi-Rothschild Theorem implies [1]. To see this, consider modified examples:

$$u_1(x,y)=\operatorname{Im}\sqrt{1+z^2}+\operatorname{Im}\left(e^{-1/\sqrt{z/i}}\right) \tag{1.7}$$

(the branch cut for $\sqrt{z}$ is on the negative y-axis), so u_1 vanishes on the y-axis and is not analytic at $z=0$, or

$$u_2(x,y)=\operatorname{Im}\sqrt{1+z^2}+\operatorname{Re}\left(e^{-1/\sqrt{z/i}}\right), \tag{1.8}$$

which does not vanish on the y-axis and is not analytic at $z=0$.

2. Domains in which (A$'$) does hold

Now consider V defined as in (1.1) and such that the y-axis is orthogonal to V, i.e.

$$y=f(x),\quad f(0)=f'(0)=0. \tag{2.1}$$

V is the image of a neighborhood of the origin on the x-axis under the map:

$$\tilde{f}(z)=z+if(z)=z+ic_{k_0}z^{k_0}+O(z^{k_0+1}). \tag{2.2}$$

Since $\tilde{f}'(0) = 1 \neq 0$, the inverse map $h(z)$ to $\tilde{f}(z)$ exists in a neighborhood of $z = 0$ and

$$h(z) = z - ic_{k_0} z^{k_0} + O(z^{k_0+1}). \tag{2.3}$$

Under this map a part of V near the origin is mapped into the x-axis while the y-axis is mapped into a real-analytic arc S defined by

$$x = \varphi(y), \quad \varphi(0) = \varphi'(0) = 0. \tag{2.4}$$

Since harmonicity of the function $u(x, y)$ is preserved under conformal maps (together with the property of vanishing to infinite order), we have the following simple

Proposition 2.1. (A$'$) *holds for a boundary arc V and a harmonic function $u(x, y)$ vanishing to infinite order on the y-axis at $z = 0$ if and only if it holds for the x-axis and for $u(\tilde{f}(z))$ vanishing to infinite order on the arc S.*

If $f(z)$ is an odd function, then $\tilde{f}(z)$ is odd and so is $h(z)$. Hence, S must be symmetric about the origin in a small neighborhood of it, i.e. $\varphi(y)$ must also be an odd function:

$$\varphi(y) = (-1)^p c y^{2p+1} + O(y^{2p+3}). \tag{2.5}$$

In a more general case, when $f(x)$ is as in (1.5) (with the same notation),

$$\varphi(y) = (-1)^p c y^{2p+1} + O(y^{2p+2}). \tag{2.6}$$

Conversely, given the arc S (by (2.4)) we can find the equation (2.1) of V using a procedure similar to the above. Again, when $\varphi(y)$ is odd or given by (2.6), then $f(x)$ is odd or given by (1.5). From Proposition 2.1 and Corollary 1.3 we obtain the following

Corollary 2.2. *Let a boundary arc V be a part of the real axis. Let $\varphi(y)$ be the defining function* (2.4) *of an arc S tangent to the imaginary axis. If the lowest non-zero term in the Taylor expansion of φ is odd* (2.6)*, then* (A$'$) *always fails for harmonic functions $u(x, y)$ vanishing to infinite order on the arc S. In particular,* (A$'$) *fails for arcs symmetric about the origin, i.e. for odd functions φ.*

The following lemma is known in connection with *Poincare's local problem of conformal geometry* (cf. [2] and, also, [3, Lemma 1]). For the reader's convenience we supply a simple proof.

Lemma 2.3. *Given an arc V (or S), we can find a conformal map sending V into the x-axis (S into the y-axis), $z = 0$ to $z = 0$ and the y-axis (x-axis) into itself if and only if V is locally symmetric about the y-axis (S is locally symmetric about the x-axis).*

Proof. It suffices to prove the lemma for the x-axis and the arc S. Suppose $\varphi(y)$ is even, i.e., $\varphi(y) = \sum_{m=1}^{\infty} c_{2m} y^{2m}$ and the points of $S: \quad z = \varphi(y) + iy$ are the images of those on the y-axis under the map $\tilde{\varphi}(z) = z + \sum_{m=1}^{\infty} (-1)^m c_{2m} z^{2m}$. Since the coefficients of $\tilde{\varphi}(z)$ are real, the inverse map $h(z)$ to $\tilde{\varphi}(z)$ also has real coefficients. Thus, the x-axis is preserved under $h(z)$ while S is mapped into the y-axis.

Conversely, suppose $h(z)$ is the required function. It preserves the real axis and so does its inverse $h^{-1}(z)$. Then S is the image of the y-axis under the map $h^{-1}(z)$.

By the Schwarz Reflection Principle $h^{-1}(\bar{z}) = \overline{h^{-1}(z)}$ and S is symmetric about the x-axis. The proof is complete.

Corollary 2.4. *If a boundary V given by* (2.1) *is symmetric with respect to the y-axis, i.e. the function $f(x)$ is even, then* (A$'$) *holds.*

Corollary 2.5. *If a boundary arc V is a part of the real axis and a function $u(x, y)$ harmonic in the domain D ($\partial D \supset V$), $u \geq 0$ on V, vanishes to infinite order on a part of an analytic arc S orthogonal to the real axis while the whole curve S is symmetric with respect to the x-axis, then u must vanish identically on some subinterval of V about the origin, i.e. the conclusion of* (A$'$) *still holds.*

Note the following obvious corollary of Proposition 2.1:

Corollary 2.6. (B$'$) *holds for any real-analytic arc V.*

Now we are ready to characterize configurations for which (A$'$) holds. Without loss of generality, consider the problem when V is a part of the x-axis and $u(x, y)$ vanishes to infinite order on an analytic arc S tangent to the y-axis at the origin. S is given by (2.4). If $\varphi(y)$ is even, i.e. its Taylor expansion near $y = 0$ contains only even powers, then Corollary 2.5 implies that (A$'$) holds. If the lowest non-zero term in the Taylor expansion of φ is odd and φ has the form (2.6), then (A$'$) fails in view of Corollary 2.2. Thus, the only remaining case is when $\varphi(y)$ has the expansion

$$\varphi(y) = c_2 y^2 + \dots + c_{2p} y^{2p} + c_{2p+1} y^{2p+1} + O(y^{2p+2}), \qquad c_{2p+1} \neq 0.$$

The curve S is the image of the y-axis under the map

$$g(z) = z - c_2 z^2 + \dots + (-1)^p c_{2p} z^{2p} + i(-1)^{p+1} c_{2p+1} z^{2p+1} + O(z^{2p+2}). \tag{2.8}$$

Set $h(z) = z - c_2 z^2 + \dots + (-1)^p c_{2p} z^{2p}$, so that

$$g(yi) = h(yi) + c_{2p+1} y^{2p+1} + O(y^{2p+2}). \tag{2.9}$$

Now let $\zeta = h^{-1}(w)$ be the inverse of h that has expansion

$$h^{-1}(w) = w + d_2 w^2 + \dots + d_{2p+1} w^{2p+1} + O(w^{2p+2}), \tag{2.10}$$

where all d_k are real. Obviously, h^{-1} sends the x-axis into itself (locally). The image of S under h^{-1} is a curve $\tilde{S}$. In view of (2.9) $\tilde{S}$ can be parametrized by

$$h^{-1}(g(yi)) = h^{-1}(h(yi) + c_{2p+1} y^{2p+1} + O(y^{2p+2})),$$

and according to (2.10) we have:

$$\begin{aligned} h^{-1}(g(yi)) &= h^{-1}(h(yi)) + c_{2p+1} y^{2p+1} + O(y^{2p+2}) \\ &= yi + c_{2p+1} y^{2p+1} + O(y^{2p+2}) \\ &= (y + O_R(y^{2p+2}))i + c_{2p+1} y^{2p+1} + O_R(y^{2p+2}). \end{aligned} \tag{2.11}$$

So in the ζ-plane, $\tilde{S}$ is parametrized by

$$\tilde{x} := \operatorname{Re}\zeta = c_{2p+1} y^{2p+1} + O_R(y^{2p+2}), \qquad \tilde{y} := \operatorname{Im}\zeta = y + O_R(y^{2p+2})$$

or, excluding y, we obtain

$$\tilde{x} = \tilde{\varphi}(\tilde{y}) = c_{2p+1} \tilde{y}^{2p+1} + O(\tilde{y}^{2p+2}), \tag{2.12}$$

i.e. $\tilde{S}$ satisfies (2.6), and (A$'$) fails because of Corollary 2.2. Applying Corollary 2.5 we now obtain

Theorem 1. *In the case when the arc V lies on the real axis and $u(x, y)$ is vanishing to infinite order on an analytic arc S tangent to the imaginary axis at the origin,* (A$'$) *holds if and only if S is locally symmetric about the x-axis, i.e. the defining function of S* (2.4) *is even. In particular, if S is a circular arc,* (A$'$) *always holds.*

Applying Proposition 2.1 and Lemma 2.3 we deduce

Theorem 2. *In the case of an arbitrary domain D with an arc V on the boundary tangent to the real axis at the origin and a function $u(x, y)$ harmonic in D and vanishing to infinite order on the imaginary axis,* (A$'$) *holds if and only if V is locally symmetric about the imaginary axis, i.e. its defining function* (1.1) *is even.*

Finally, for the most general configuration of a boundary arc $V \subset \partial D$ and an analytic arc $S \subset D$ orthogonal to V we have

Theorem 3. *For a harmonic function $u(x, y)$ in D vanishing to infinite order on the arc S and non-negative on a boundary arc V,* (A$'$) *holds if and only if there is a conformal map that sends V and S respectively into the real and imaginary axes. In particular, if S and V are mutually orthogonal circular arcs,* (A$'$) *holds.*

Corollary 2.7 ("Conjugacy principle"). *For the configuration of a boundary arc $V \subset \partial D$ and harmonic functions in D vanishing to infinite order on an analytic arc S orthogonal to V,* (A$'$) *holds if and only if it holds for the "dual configuration" of the domain D': $\partial D' \supset S$ and harmonic functions in D' vanishing to infinite order on the arc V.*

Also, note that if S is a C^∞-curve such that its defining function $\varphi(y)$ vanishes to infinite order at $y = 0$, then (A$'$) obviously holds for $V \subset \mathbf{R}$ and harmonic functions u vanishing to infinite order on S (the proof is similar to that in [1]).

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