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ON THE HIGHER DELTA INVARIANTS OF A GORENSTEIN LOCAL RING

YUJI YOSHINO

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To the memory of Professor Maurice Auslander

ABSTRACT. Let $(R, \mathfrak{m})$ be a Gorenstein complete local ring. Auslander's higher delta invariants are denoted by $\delta_R^n(M)$ for each module M and for each integer n. We propose a conjecture asking if $\delta_R^n(R/\mathfrak{m}^\ell) = 0$ for any positive integers n and ℓ. We prove that this is true provided the associated graded ring of R has depth not less than $\dim R - 1$. Furthermore we show that there are only finitely many possibilities for a pair of positive integers (n, ℓ) for which $\delta_R^n(R/\mathfrak{m}^\ell) > 0$.

0. INTRODUCTION

Let $(R, \mathfrak{m}, k)$ be a commutative Noetherian Gorenstein complete local ring. Maurice Auslander has introduced the delta invariant $\delta_R(M)$ for a finitely generated R-module M. It is defined to be the smallest integer μ such that there is an epimorphism $X \oplus R^\mu \longrightarrow M$ with X a maximal Cohen-Macaulay module with no free summands. For an integer $n \geqq 0$, the nth delta invariant $\delta_R^n(M)$ is also defined by Auslander, Ding and Solberg [2] as $\delta_R^n(M) = \delta_R(\Omega_R^n(M))$, where $\Omega_R^n(M)$ denotes the nth syzygy module of M over R. On the other hand, S. Ding ([3], [4] and [5]) studies the delta invariants of $R/\mathfrak{m}^\ell$ ($\ell \geqq 1$) and defines an interesting, new invariant - the index of R.

In this paper we are particularly interested in the higher delta invariants of $R/\mathfrak{m}^\ell$, that is, $\delta_R^n(R/\mathfrak{m}^\ell)$ for $n, \ell \geqq 1$. We would like to propose the following conjecture:

Conjecture (0.1). *For any positive integers n and ℓ, we would have $\delta_R^n(R/\mathfrak{m}^\ell) = 0$ unless R is regular.*

Actually, as one of the main theorems of this paper, we shall show this conjecture is true if the associated graded ring $gr_{\mathfrak{m}}(R)$ has depth $\geqq \dim R - 1$. See Corollary (2.3). In particular, it is valid if R is either a ring of hypersurface or a homogeneous graded ring. We note that, if $d = \dim(R)$, then that $\delta_R^d(R/\mathfrak{m}^\ell) = 0$ exactly means that the dth syzygy module of $R/\mathfrak{m}^\ell$ has no free direct summand. Hence our result generalizes a result of Herzog [6, Cor (2.4)].

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Furthermore we can show in Theorem (2.1) that, in general, there are only finitely many possibilities for a pair (n, ℓ) of positive integers for which $\delta_R^n(R/\mathfrak{m}^\ell) > 0$.

1. Some preliminaries

In this paper $(R, \mathfrak{m}, k)$ will always be a Gorenstein complete local ring. We recall some basic facts on the delta invariants from [1] and [2].

For a finitely generated R-module M, an exact sequence

$$0 \longrightarrow Y \xrightarrow{f} X \longrightarrow M \longrightarrow 0 \tag{1.1}$$

is called a Cohen-Macaulay approximation of M if X is a maximal Cohen-Macaulay module and Y has finite projective dimension. We say that the sequence is minimal if there are no isomorphisms split out of f. It is known that a minimal Cohen-Macaulay approximation of M exists uniquely up to isomorphisms.

An R-module is said to be stable if there is no nontrivial free direct summand. Since R is a complete local ring, every finitely generated R-module X is uniquely decomposed as a direct sum of a stable module and a free module. We denote the maximal rank of free direct summand of X by $\text{f-rank}_R X$. If the sequence (1.1) is the minimal Cohen-Macaulay approximation of M, then the delta invariant $\delta_R(M)$ is defined as $\text{f-rank}_R X$. It is known that $\delta_R(M) = 0$ if and only if M is a homomorphic image of a stable maximal Cohen-Macaulay module.

For an integer n, the nth delta invariant $\delta_R^n(M)$ is, by definition, the delta invariant of the nth syzygy module $\Omega_R^n(M)$ of M. Note that if $n > \dim R$, then $\delta_R^n(M) = 0$ for any finitely generated module M, since $\Omega_R^n(M)$ is a stable maximal Cohen-Macaulay module.

Auslander has shown the following

Lemma (1.2) ([2, Proposition 5.7]). *If R is non-regular, then $\delta_R^n(k) = 0$ for any $n \geqq 0$.*

As a result of this lemma we have

Lemma (1.3). *Suppose that R is non-regular and let M be a finitely generated R-module. Then $\delta_R(\mathfrak{m}M) = 0$. In particular, we see that $\delta_R^1(R/\mathfrak{m}^\ell) = 0$ for any ℓ.*

Proof. Since $\delta_R(\mathfrak{m}) = \delta_R^1(R/\mathfrak{m}) = 0$, there is an epimorphism $X \longrightarrow \mathfrak{m}$ with X a stable maximal Cohen-Macaulay module. If $F \longrightarrow M$ is a free cover of M, then we have an epimorphism $F \otimes_R X \longrightarrow \mathfrak{m}M$, where $F \otimes_R X$ is also a stable maximal Cohen-Macaulay module. Thus $\delta_R(\mathfrak{m}M) = 0$. □

Corollary (1.4). *Conjecture* (0.1) *is true if* $\dim R \leqq 1$.

We now remark on delta invariants under some ring extension.

Lemma (1.5). *Let $\varphi : R \longrightarrow R'$ be a local homomorphism of Gorenstein complete local rings and let M be a finitely generated R-module. Assume that φ is a faithfully flat morphism and* $\dim R = \dim R'$. *Let the minimal Cohen-Macaulay approximation of M over R be given as* (1.1). *Then the sequence*

$$0 \longrightarrow Y \otimes_R R' \xrightarrow{f \otimes R'} X \otimes_R R' \longrightarrow M \otimes_R R' \longrightarrow 0 \tag{1.6}$$

is the minimal Cohen-Macaulay approximation over R'.

Proof. Since R' is R-flat and has the same dimension as R, we can show that if X is a maximal Cohen-Macaulay module over R (resp. has finite projective dimension over R), then so is $X \otimes_R R'$ over R'. Thus the sequence (1.6) is a Cohen-Macaulay approximation of $M \otimes_R R'$ over R'. We have to show that (1.6) is minimal. Suppose not. Then we would have an R'-homomorphism $g : X \otimes_R R' \longrightarrow R'$ such that the composition $g \cdot (f \otimes R')$ is an epimorphism. Since $\mathrm{Hom}_{R'}(X \otimes_R R', R') \cong \mathrm{Hom}_R(X, R) \otimes_R R'$, we can write g as a finite sum of $h_i \otimes R'$ with $h_i \in \mathrm{Hom}_R(X, R)$. Since R and R' are local rings, there is an i such that $(h_i \cdot f) \otimes R'$, hence $h_i \cdot f$, is an epimorphism. This contradicts the sequence (1.1) being minimal. □

Lemma (1.7). *Under the same notation as in* (1.5)*, we have the equality*

$$\delta_R^n(M) = \delta_{R'}^n(M \otimes_R R'),$$

for any integer $n \geqq 0$.

Proof. Since R' is R-flat, we can see that $\Omega_{R'}^n(M \otimes_R R') = \Omega_R^n(M) \otimes_R R'$. Thus applying Lemma (1.5), we have only to show that $\text{f-rank}_R(X) = \text{f-rank}_{R'}(X \otimes_R R')$ if X is a maximal Cohen-Macaulay module over R. For this, it is enough to prove that if X is a stable R-module, then $X \otimes_R R'$ is also stable as an R'-module. Suppose $X \otimes_R R'$ is not stable over R'. Then we have an epimorphism $g : X \otimes_R R' \longrightarrow R'$. As in the proof of (1.5) we can write g as a finite sum of $h_i \otimes R'$ with $h_i \in \mathrm{Hom}_R(X, R)$. Then h_i is an epimorphism for some i. This contradicts that X is stable. □

The same equality as in (1.7) is discussed in a recent work of Shida [7]. The following lemma is necessary for the proof of our theorem:

Lemma (1.8). *Let* $x \in \mathfrak{m}$ *be a non zero-divisor both on* R *and on a finitely generated* R*-module* M*. Putting* $\overline{R} = R/xR$*, we have the equality* $\delta_{\overline{R}}(M \otimes_R \overline{R}) = \delta_R(M)$.

This lemma follows easily from [2, Lemma 5.1].

2. Main theorems

In the rest of the paper, $(R, \mathfrak{m}, k)$ is a complete Gorenstein local ring of dimension d, and **we always assume that** R **is non-regular**. Our main theorems in this paper are the following:

Theorem (2.1). *There is an integer* ℓ_0 *such that* $\delta_R^n(R/\mathfrak{m}^\ell) = 0$ *for any* $\ell \geqq \ell_0$ *and for any* $n > 0$.

Theorem (2.2). *Let* $G = gr_{\mathfrak{m}}(R)$ *be the associated graded ring of* R *with respect to the maximal ideal. If* $\operatorname{depth} G = t$*, then we have* $\delta_R^n(R/\mathfrak{m}^\ell) = 0$ *for any* $\ell > 0$ *and for any* $n \geqq d - t + 1$.

As a consequence of the above theorems, we see that there are only finitely many possibilities for a pair of positive integers (n, ℓ) with $\delta_R^n(R/\mathfrak{m}^\ell) > 0$. Furthermore we have the following corollary as a direct consequence of (2.2), which generalizes a result of Herzog [6, Cor. (2.4)].

Corollary (2.3). *Suppose that the associated graded ring* G *has depth* $\geqq d - 1$*. (For example,* R *is either a ring of hypersurface or a homogeneous graded ring.) Then* $\delta_R^\ell(R/\mathfrak{m}^n) = 0$ *for any positive integers* n *and* ℓ.

Proof. Apply (2.2) to get $\delta_R^n(R/\mathfrak{m}^\ell) = 0$ for $n \geqq 2$, then combine it with (1.3). □

Before proceeding to the proof, we make the following

Remark 2.4. When proving Theorems (2.1) and (2.2), we may assume that the residue field k is an infinite field.

To show this, let u be an indeterminate and let R' be the completion of the local ring $R[u]_{\mathfrak{m}[u]}$. Then R' is a faithfully flat R-algebra, which is a complete Gorenstein local ring with maximal ideal $\mathfrak{m}' = \mathfrak{m}R'$ and with residue field $k' = k(u)$ that is, in fact, an infinite field. Notice in this setting that $R'/\mathfrak{m}'^{\ell} = R/\mathfrak{m}^{\ell} \otimes_R R'$. Hence it follows from (1.7) that $\delta^n_{R'}(R'/\mathfrak{m}'^{\ell}) = \delta^n_R(R/\mathfrak{m}^{\ell})$. Note also that $gr_{\mathfrak{m}'}(R') = gr_{\mathfrak{m}}(R) \otimes_R R'$, in particular, one can show $\operatorname{depth} gr_{\mathfrak{m}'}(R') = \operatorname{depth} gr_{\mathfrak{m}}(R)$. Thus, if necessary, taking R' instead of R, we may assume that R has an infinite residue field. □

We need a lemma to prove the theorems.

Lemma (2.5). *Let M be a finitely generated R-module and let $x \in \mathfrak{m}$ be a non-zero-divisor on R. Suppose that the initial form x^* of x in $G = gr_{\mathfrak{m}}(R)$ is a non-zero divisor on $gr_{\mathfrak{m}}(M)$. Furthermore we denote $\overline{R} = R/xR$ and $\overline{M} = M/xM$. Then we have the following isomorphism for each $n \geqq 0$:*

$$\Omega^n_R(\mathfrak{m}M) \otimes_R \overline{R} \cong \Omega^n_{\overline{R}}(\mathfrak{m}\overline{M}) \oplus \Omega^n_{\overline{R}}(M/\mathfrak{m}M).$$

Proof. We note from the assumption that x is a non-zero divisor on M and the multiplication map $M/\mathfrak{m}M \to \mathfrak{m}M/\mathfrak{m}^2M$ of x is an injective map. Thus we have the following commutative diagram with an exact row:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & M/\mathfrak{m}M & \xrightarrow{x} & \mathfrak{m}M/x\mathfrak{m}M & \longrightarrow & \mathfrak{m}M/xM & \longrightarrow & 0 \\ & & \big\downarrow{=} & & \big\downarrow{\text{natural map}} & & & & \\ & & M/\mathfrak{m}M & \xrightarrow{x} & \mathfrak{m}M/\mathfrak{m}^2M & & & & \end{array}$$

As the map in the second row is an injective map of k-vector spaces, it is split and thus the first row is a split exact sequence. Therefore we see that

$$\mathfrak{m}M \otimes_R \overline{R} \cong \mathfrak{m}(M/xM) \oplus M/\mathfrak{m}M.$$

Now we prove the lemma by induction on n. For $n = 0$ the lemma is nothing but the above isomorphism. For $n > 0$, we take the minimal free cover of $\Omega^{n-1}_R(\mathfrak{m}M)$ to get the exact sequence:

$$0 \longrightarrow \Omega^n_R(\mathfrak{m}M) \longrightarrow F \longrightarrow \Omega^{n-1}_R(\mathfrak{m}M) \longrightarrow 0.$$

Taking the tensor product of this sequence with $\overline{R}$, we can show that the sequence

$$0 \longrightarrow \Omega^n_R(\mathfrak{m}M) \otimes_R \overline{R} \longrightarrow F \otimes_R \overline{R} \longrightarrow \Omega^{n-1}_R(\mathfrak{m}M) \otimes_R \overline{R} \longrightarrow 0$$

gives a minimal free cover over $\overline{R}$. Thus we have an isomorphism

$$\Omega^n_R(\mathfrak{m}M) \otimes_R \overline{R} \cong \Omega^1_{\overline{R}}\left(\Omega^{n-1}_R(\mathfrak{m}M) \otimes_R \overline{R}\right).$$

The lemma follows from this isomorphism together with the induction hypothesis. □

We should mention that a reduction argument similar to that used in this lemma already appeared in Ding [5].

Proof of Theorem (2.1). In this proof we assume that k is an infinite field by (2.4). We prove the theorem by induction on $d = \dim R$. If $d \leqq 1$, then the theorem holds by (1.4).

Now suppose $d > 1$ and we denote the associated graded ring of R with respect to the maximal ideal by $G = gr_{\mathfrak{m}}(R)$ and the irrelevant maximal ideal of G by G_+. Let $I = \mathrm{H}^0_{G_+}(G)$ be the ideal of G consisting of all elements which are annihilated by some powers of G_+. Note that I is a graded ideal of finite length, hence there is an integer ℓ_1 such that $I_\ell = (0)$ for any $\ell \geqq \ell_1$. Since G/I has positive depth and since k is an infinite field, we can find an element $x \in \mathfrak{m} - \mathfrak{m}^2$ so that the initial form $x^* \in G$ is a non-zero divisor on G/I. Then, for $\ell \geqq \ell_1$, the multiplication of x induces the injective map

$$\mathfrak{m}^\ell/\mathfrak{m}^{\ell+1} \longrightarrow \mathfrak{m}^{\ell+1}/\mathfrak{m}^{\ell+2},$$

since $(G/I)_\ell = G_\ell$. In particular, one sees that x^* is a non-zero-divisor on the G-module $gr_{\mathfrak{m}}(\mathfrak{m}^\ell)$ for $\ell \geqq \ell_1$. Thus we can apply Lemma (2.5) to get the isomorphism:

$$\Omega^n_R(\mathfrak{m}^{\ell+1}) \otimes_R \overline{R} \cong \Omega^n_{\overline{R}}(\mathfrak{m}^{\ell+1}/x\mathfrak{m}^\ell) \oplus \Omega^n_{\overline{R}}(\mathfrak{m}^\ell/\mathfrak{m}^{\ell+1}) \quad (\ell \geqq \ell_1), \tag{2.6}$$

where $\overline{R} = R/xR$.

Let $\overline{\mathfrak{m}}$ be the maximal ideal $\mathfrak{m}/xR$ of $\overline{R}$. We shall show that there is an integer $\ell_2 (\geqq \ell_1)$ such that $\overline{\mathfrak{m}}^{\ell+1} = \mathfrak{m}^{\ell+1}/x\mathfrak{m}^\ell$ for any $\ell \geqq \ell_2$. To show this, it is enough to prove $xR \cap \mathfrak{m}^{\ell+1} = x\mathfrak{m}^\ell$ for $\ell \geqq \ell_2$, since $\overline{\mathfrak{m}}^{\ell+1} = \mathfrak{m}^{\ell+1}/xR \cap \mathfrak{m}^{\ell+1}$. By the lemma of Artin-Rees, we know that there is an integer r (> 0) such that $xR \cap \mathfrak{m}^{\ell+1} = \mathfrak{m}^{\ell+1-r}(xR \cap \mathfrak{m}^r)$ for $\ell \geqq r$. Hence

$$xR \cap \mathfrak{m}^{\ell+1} = x\mathfrak{m}^{\ell+1-r} \cap \mathfrak{m}^{\ell+1}, \tag{2.7}$$

for such ℓ. Now take ℓ such that $\ell \geqq \ell_1 + r - 1$. Then, since $\ell + i - r \geqq \ell_1$ for $i \geqq 1$, we see that $(G/I)_{\ell+i-r} = G_{\ell+i-r}$ for $i \geqq 1$. Thus the multiplication by x induces the following sequence of injective maps:

$$\mathfrak{m}^{\ell+1-r}/\mathfrak{m}^{\ell+2-r} \longrightarrow \mathfrak{m}^{\ell+2-r}/\mathfrak{m}^{\ell+3-r} \longrightarrow \cdots \longrightarrow \mathfrak{m}^{\ell-1}/\mathfrak{m}^\ell \longrightarrow \mathfrak{m}^\ell/\mathfrak{m}^{\ell+1}.$$

This means that the right hand side of (2.7) equals $x\mathfrak{m}^\ell$. Therefore we have shown the equality $\overline{\mathfrak{m}}^{\ell+1} = \mathfrak{m}^{\ell+1}/x\mathfrak{m}^\ell$ holds for $\ell \geqq \ell_2 := \ell_1 + r - 1$. Combining this with (2.6), we thus have

$$\Omega^n_R(\mathfrak{m}^{\ell+1}) \otimes_R \overline{R} \cong \Omega^n_{\overline{R}}(\overline{\mathfrak{m}}^{\ell+1}) \oplus \Omega^n_{\overline{R}}(\mathfrak{m}^\ell/\mathfrak{m}^{\ell+1}), \tag{2.8}$$

for $\ell \geqq \ell_2$.

Now by the induction hypothesis, we find an integer ℓ_3 such that

$$\delta_{\overline{R}}\left(\Omega^n_{\overline{R}}(\overline{\mathfrak{m}}^{\ell+1})\right) = \delta^n_{\overline{R}}(\overline{\mathfrak{m}}^{\ell+1}) = 0,$$

for $\ell \geqq \ell_3$. On the other hand, we know from (1.2) that $\delta^n_{\overline{R}}(\mathfrak{m}^\ell/\mathfrak{m}^{\ell+1}) = 0$ for any ℓ and n. Thus it follows from (2.8) that $\delta_{\overline{R}}\left(\Omega^n_R(\mathfrak{m}^{\ell+1}) \otimes_R \overline{R}\right) = 0$ for $\ell \geqq \ell_0 := \max\{\ell_2, \ell_3\}$. Then it follows from (1.8) that $\delta^n_R(\mathfrak{m}^{\ell+1}) = \delta_R\left(\Omega^n_R(\mathfrak{m}^{\ell+1})\right) = 0$ for $\ell \geqq \ell_0$ as desired. □

Proof of Theorem (2.2). The proof of this theorem goes through in the same way as the proof of (2.1). We may assume that k is an infinite field. We prove the theorem by induction on $t = \operatorname{depth} G$. If $t = 0$, then the theorem obviously holds since $\delta^n_R(M) = 0$ for any $n > d = \dim R$ and for any finitely generated R-module M. Assume $t \geqq 1$. Since k is infinite, there is an element $x \in \mathfrak{m} - \mathfrak{m}^2$ such that the initial form $x^* \in G$ is a non-zero-divisor on G. For a fixed integer ℓ, we can apply

Lemma (2.5) to $M = \mathfrak{m}^{\ell-1}$, since x^* is also a non-zero-divisor on $gr_{\mathfrak{m}}(\mathfrak{m}^{\ell-1}) \subseteq G$. Thus we have an isomorphism for each $n > 0$:

$$\Omega_R^{n-1}(\mathfrak{m}^\ell) \otimes_R \overline{R} \cong \Omega_{\overline{R}}^{n-1}(\overline{\mathfrak{m}}^\ell) \oplus \Omega_{\overline{R}}^{n-1}(\mathfrak{m}^{\ell-1}/\mathfrak{m}^\ell), \tag{2.9}$$

where $\overline{R} = R/xR$ and $\overline{\mathfrak{m}} = \mathfrak{m}/xR$. Note in this case that we have $\mathfrak{m}^{\ell+1}/x\mathfrak{m}^\ell = \overline{\mathfrak{m}}^{\ell+1}$ for any $\ell \geqq 0$, since x^* is a non-zero divisor on G. Compare with the proof of (2.1). We also notice that $gr_{\overline{\mathfrak{m}}}(\overline{R}) = G/x^*G$, in particular, $\operatorname{depth} gr_{\overline{\mathfrak{m}}}(\overline{R}) = t-1$. Thus by the induction hypothesis, we see that

$$\delta_{\overline{R}}\left(\Omega_{\overline{R}}^{n-1}(\overline{\mathfrak{m}}^\ell)\right) = \delta_{\overline{R}}\left(\Omega_{\overline{R}}^{n}(\overline{R}/\overline{\mathfrak{m}}^\ell)\right) = \delta_{\overline{R}}^{n}\left(\overline{R}/\overline{\mathfrak{m}}^\ell\right) = 0,$$

for any $n \geqq (d-1)-(t-1)+1 = d-t+1$. On the other hand, we know from (1.2) that $\delta_{\overline{R}}\left(\Omega_{\overline{R}}^{n-1}(\mathfrak{m}^{\ell-1}/\mathfrak{m}^\ell)\right) = \delta_{\overline{R}}^{n-1}(\mathfrak{m}^{\ell-1}/\mathfrak{m}^\ell) = 0$ for any $n > 0$. Therefore it follows from (2.9) that $\delta_{\overline{R}}\left(\Omega_R^{n-1}(\mathfrak{m}^\ell) \otimes_R \overline{R}\right) = 0$ if $n \geqq d-t+1$. Thus Lemma (1.8) implies that $\delta_R^n(R/\mathfrak{m}^\ell) = \delta_R^{n-1}(\mathfrak{m}^\ell) = \delta_R\left(\Omega_R^{n-1}(\mathfrak{m}^\ell)\right) = 0$ as desired. □

3. Some remarks for two-dimensional cases

In this section we assume that the Gorenstein complete local ring $(R, \mathfrak{m}, k)$ has dimension 2. We denote the associated graded ring $gr_{\mathfrak{m}}(R)$ by G, as before.

As we have shown in (2.2), if G has positive depth, then the conjecture (0.1) is true for R. However there is, of course, a Gorenstein local ring R with $\operatorname{depth} G = 0$. One of the easiest examples is

$$R = k[[u, v, w, x, y]]/(u^2, ux - v^3, uy - w^3),$$

for which I do not know if the conjecture is true or not.

If one wants to make a counterexample to the conjecture (0.1), the following lemma will be useful.

Lemma (3.1). *The following two conditions are equivalent for an $\mathfrak{m}$-primary ideal I.*

(a) $\delta_R^2(R/I) > 0$.

(b) *There is an exact sequence*

$$0 \longrightarrow R \xrightarrow{\ j\ } L \xrightarrow{\ p\ } I \longrightarrow 0$$

with $p \otimes_R k$ an isomorphism (or equivalently $j(1) \in \mathfrak{m}L$).

Proof. (a) $\Longrightarrow$ (b) Since $M := \Omega^2(R/I)$ is a maximal Cohen-Macaulay module over R, the condition (a) says exactly that M contains a free module as a direct summand. Thus there is an epimorphism $\rho : M \longrightarrow R$ and we have a push-out diagram

$$\begin{array}{ccccccccc}
0 & \longrightarrow & M & \longrightarrow & F & \xrightarrow{\ \pi\ } & I & \longrightarrow & 0 \\
 & & \rho\downarrow & & \phi\downarrow & & =\downarrow & & \\
0 & \longrightarrow & R & \xrightarrow{\ j\ } & L & \xrightarrow{\ p\ } & I & \longrightarrow & 0,
\end{array}$$

where π is a minimal free cover of I. Since $\pi = p \cdot \phi$, we see that $(p \otimes k) \cdot (\phi \otimes k) = \pi \otimes k$ is an isomorphism. Note from the diagram that $\phi \otimes k$ is an epimorphism and we have that $p \otimes k$ is an isomorphism.

$(b) \Longrightarrow (a)$ Let $\phi : F \longrightarrow L$ be a minimal free cover of L. Then we have the commutative diagram:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & X := \mathrm{Ker}(p\cdot\phi) & \longrightarrow & F & \xrightarrow{p\cdot\phi} & I & \longrightarrow & 0 \\ & & \psi\big\downarrow & & \phi\big\downarrow & & =\big\downarrow & & \\ 0 & \longrightarrow & R & \xrightarrow{j} & L & \xrightarrow{p} & I & \longrightarrow & 0 \end{array}$$

Since $(p \cdot \phi) \otimes k = (p \otimes k) \cdot (\phi \otimes k)$ is an isomorphism, we can see that $p \cdot \phi$ is a minimal free cover of I. Thus we have $X \cong \Omega_R^2(R/I)$. On the other hand, it follows from the diagram that ψ is an epimorphism and that $\Omega_R^2(R/I)$ has R as a direct summand. □

As one of the consequences of (3.1), we get the following

Proposition (3.2). *Suppose that there exist an integer r and a system of parameters $\{x, y\}$ of R which satisfy the following conditions:*
(*a*) $(x, y) \subseteqq \mathfrak{m}^{r+1}$.
(*b*) $(x, y)\mathfrak{m}^r = \mathfrak{m}^{2r+1}$.
Then we have $\delta_R^2(R/\mathfrak{m}^{2r+1}) > 0$.

Proof. Let $I = (x, y)R$ and take the free resolution of I:

$$0 \longrightarrow R \xrightarrow{f_2} R^2 \xrightarrow{f_1} I \longrightarrow 0,$$

which is part of the Koszul complex. Then we see that $\mathrm{Im}(f_2) \subseteqq IR^2 \subseteqq \mathfrak{m}^{r+1}R^2$. Thus we have the exact sequence

$$0 \longrightarrow R \xrightarrow{f_2} \mathfrak{m}^r R^2 \xrightarrow{f_1'} \mathfrak{m}^r I \longrightarrow 0,$$

where f_1' is the restriction to f_1 on $\mathfrak{m}^r R^2$. Since $f_2(1) \in \mathfrak{m}\left(\mathfrak{m}^r R^2\right)$, it follows from (3.1) that $\delta_R^2(R/\mathfrak{m}^r I) > 0$. □

If R is a regular local ring, then the regular system of parameters satisfies the conditions in (3.2) for $r = 0$. For a non-regular ring R, if there is a system of parameters satisfying the conditions (a), (b) in (3.4), then we must have depth $G = 0$ by (2.2). We do not know if there is a non-regular Gorenstein local ring with the conditions in (3.4), or not.

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Institute of Mathematics, Faculty of Integrated Human Studies, Kyoto University, Yoshida-Nihonmatsu, Sakyo-ku, Kyoto 606-01, Japan

E-mail address: yoshino@math.h.kyoto-u.ac.jp