PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 124, Number 9, September 1996

OSCILLATORY SINGULAR INTEGRALS ON L^p AND HARDY SPACES

YIBIAO PAN

(Communicated by J. Marshall Ash)

Abstract. We consider boundedness properties of oscillatory singular integrals on L^p and Hardy spaces. By constructing a phase function, we prove that H^1 boundedness may fail while L^p boundedness holds for all $p \in (1, \infty)$. This shows that the L^p theory and H^1 theory for such operators are fundamentally different.

§1. Introduction

Let Φ be a real-valued C^1 function on $[-1, 1]$, $\Phi(0) = \Phi'(0) = 0$, $\lambda \in \mathbf{R}$, and define T_λ by

$$T_\lambda f(x) = \text{p.v.} \int_{|x-y| \le 1} e^{i\lambda\Phi(x-y)} \frac{f(y)}{x-y} f(y) dy. \tag{1}$$

Such operators are called oscillatory singular integral operators and have been studied extensively ([2], [3], [6], [9], [10], [12]). If Φ is sufficiently smooth and $\Phi^{(k)}(0) \neq 0$ for some $k > 1$, then the L^p and H^1 boundedness of T_λ is well-known ([13], [6], [7]).

Theorem A. *Suppose Φ is sufficiently smooth and $\Phi^{(k)}(0) \neq 0$ for some $k > 1$; then T_λ are uniformly bounded on $L^p(\mathbf{R})$ $(1 < p < \infty)$ and $H^1(\mathbf{R})$.*

The space $H^1(\mathbf{R})$ is the usual Hardy space H^1.

Both the L^p and H^1 uniform boundedness of $\{T_\lambda\}_{\lambda \in \mathbf{R}}$ may fail if $\Phi^{(n)}(0) = 0$ for all n ([5], [7]). On the other hand, the condition that $\Phi^{(k)}(0) \neq 0$ for some $k > 1$ is not a necessary condition. By using results of Nagel et al. ([4]) on Hilbert transforms along curves, one can obtain the following L^2 result:

Theorem B. *Suppose Φ is even, $\Phi(0) = \Phi'(0) = 0$ and $\Phi'' > 0$. Then the operators T_λ are uniformly bounded on $L^2(\mathbf{R})$ if and only if there is a $C > 0$ such that*

$$\Phi'(Ct) > 2\Phi'(t), \tag{2}$$

for every $t > 0$.

More recent results due to Carlsson et al. ([1]) imply that the uniform L^p boundedness of T_λ holds under exactly the same condition.

Received by the editors November 15, 1994 and, in revised form, March 25, 1995.
1991 *Mathematics Subject Classification.* Primary 42B20.

Theorem C. *Suppose Φ is given as in Theorem B and $p \in (1, \infty)$. Then T_λ are uniformly bounded on L^p if and only if Φ satisfies (2).*

Results concerning odd Φ can be found in [4] and [1]. For the uniform H^1 boundedness of T_λ we have the following ([8]):

Theorem D. *Suppose* $\Phi(0) = \Phi'(0) = 0$ *and*

$$\Phi'''(t) \geq 0, \tag{3}$$

for $t > 0$. Then T_λ are uniformly bounded on $H^1(\mathbf{R})$.

To give an example of Φ which has vanishing derivatives at $t = 0$ but satisfies (2) and (3), we cite the function $\Phi(t) = e^{-4/t^2}$. We point out that condition (3) is strictly stronger than condition (2). By using interpolation and Theorem B, it is easy to see that (2) is a necessary condition for the uniform H^1 boundedness of T_λ (when Φ is even and convex). In light of Theorem C, it seems reasonable to speculate that (2) may also be a sufficient condition for the uniform H^1 boundedness of T_λ. This turns out to be false. The purpose of this paper is to construct an even, convex function Φ which satisfies (2) such that the corresponding T_λ's are not uniformly bounded on $H^1(\mathbf{R})$.

Theorem E. *There exists a function Φ which is even and satisfies (i) $\Phi(0) = \Phi'(0) = 0$ and (ii) $\Phi'' > 0$ such that*

$$\sup_{\lambda \in \mathbf{R}} \|T_\lambda\|_{L^p \to L^p} < \infty, \tag{4}$$

for $1 < p < \infty$, and

$$\sup_{\lambda \in \mathbf{R}} \|T_\lambda\|_{H^1 \to H^1} = \infty. \tag{5}$$

§2. Construction of Φ

For $k \leq -1$, we choose a function $P_k(t)$ on $[2^{k+1} - 2^k/8, 2^{k+1}]$ such that

(i) $P_k(2^{k+1} - 2^k/8) = 2^{2k} + 7 \cdot 2^{5k-2}$, $P_k(2^{k+1}) = 2^{2k+2}$;

(ii) $P_k'(2^{k+1} - 2^k/8) = 2^{4k+1}$, $P_k'(2^{k+1}) = 2^{4k+5}$;

(iii) $0 < P_k'(t) < C2^k$, for $t \in [2^{k+1} - 2^k/8, 2^{k+1}]$, where C is some positive constant.

Define g on $[0,1]$ by $g(0) = 0$ and $g(t) = 2^{2k} + 2^{4k+1}(t - 2^k)$ if $t \in [2^k, 2^{k+1} - 2^k/8)$; $g(t) = P_k(t)$ if $t \in [2^{k+1} - 2^k/8, 2^{k+1}]$. Clearly we have $g \in C^1([0,1])$. Define $\Phi(t)$ for $t \geq 0$ by

$$\Phi(t) = \int_0^t g(s)ds.$$

For $t \leq 0$ we let $\Phi(t) = \Phi(-t)$. Therefore we have $\Phi \in C^2([-1,1])$, $\Phi(0) = \Phi'(0) = 0$, and $\Phi''(t) > 0$ for $t \neq 0$. Let T_λ be given by (1). The following result can be found in [14].

Lemma 1. *Let $\{n_j\}$ be a sequence of positive integers such that $n_{j+1}/n_j \geq \sigma > 1$ for $j = 1, 2, \dots$, and let $\{\xi_j, \gamma_j\}$ be a sequence of pairs of real numbers. If $\sum_{j=1}^{\infty}(\xi_j^2 + \eta_j^2) < \infty$, then there is a function $f \in C([0, 2\pi])$ such that*

$$\int_0^{2\pi} f(t) \cos n_j t dt = \xi_j,$$

and

$$\int_0^{2\pi} f(t)\sin n_j t dt = \gamma_j,$$

for $j = 1, 2, \ldots$.

By Lemma 1, there exists a function $\eta \in C([0, 2\pi])$ such that

$$\int_0^{2\pi} \cos(2^j t)\eta(t)dt = 0; \tag{6}$$

and

$$\int_0^{2\pi} \sin(2^j t)\eta(t)dt = 1/j, \tag{7}$$

for $j = 1, 2, \ldots$.

Define $b(t)$ on $\mathbf{R}$ by $b(t) = \eta(t)$ if $t \in [0, 2\pi]$, $b(t) = -\eta(-t)$ if $t \in [-2\pi, 0)$, and $b(t) \equiv 0$ if $t \notin [-2\pi, 2\pi]$. We have $b \in H^1(\mathbf{R})$ and $\|b\|_{H^1} = c_0 > 0$.

Proposition 1. *Let* Φ *be defined as above. Then for* $p \in (1, \infty)$*, there exists* $C_p > 0$ *such that*

$$\sup_{\lambda \in \mathbf{R}} \|T_\lambda\|_{L^p \to L^p} \le C_p. \tag{8}$$

Proof. Since $2^{2k} \le \Phi'(t) \le 2^{2k+2}$ for $t \in [2^k, 2^{k+1}]$, we have

$$\Phi'(8t) > 2\Phi'(t)$$

for $t \ge 0$. Therefore (8) follows from Theorem C. □

Proposition 2. *Let* $a(x)$ *be a function defined on* $[-\delta, \delta]$ *and satisfying* $\|a\|_\infty \le (2\delta)^{-1}$ *and*

$$\int_{-\delta}^{\delta} a(x)dx = 0.$$

Then

$$\int_{|x|>2\delta} \frac{1}{|x|} \left| \int_{-\delta}^{\delta} e^{i\lambda\Phi(x-y)} a(y)dy \right| dx \le \|T_\lambda a\|_1 + C, \tag{9}$$

for some C *which is independent of* a *and* λ.

Proof.

$$\|T_\lambda a\|_1 \ge -\int_{|x|\le 2\delta} |T_\lambda a(x)|dx$$

$$-\int_{|x|>2\delta}\int_{-\delta}^{\delta} \left|\frac{1}{x} - \frac{1}{x-y}\right| |a(y)|dydx + \int_{|x|>2\delta} \frac{1}{|x|} \left|\int_{-\delta}^{\delta} e^{i\lambda\Phi(x-y)}a(y)dy\right| dx$$

$$= -I_1 - I_2 + I_3.$$

By Proposition 1 and Hölder's inequality, we have

$$I_1 \le C\delta^{1/2}\|T_\lambda a\|_2 \le C.$$

For I_2 we have

$$I_2 \leq \int_{2\delta}^{\infty} \frac{\delta^2 \|a\|_\infty}{x^2} dx \leq C.$$

Therefore, (9) holds. $\square$

We now prove Theorem E. Let N be a large integer and $\lambda = 2^{N+5}\pi$. Define $a_N(x)$ by

$$a_N(x) = (2^{N/3+4}\pi) b(2^{N/3+4}\pi x).$$

Therefore we have $supp(a_N) \subset [-2^{-N/3-3}, 2^{-N/3-3}]$, and $\|a_N\|_{H^1} \geq c_0 > 0$ for some c_0 independent of N.

For $k \leq -1$, $t \in [2^k, 2^{k+1} - 2^k/8]$ we have

$$\Phi(t) = \Phi(2^k) + 2^{2k}(t - 2^k) + 2^{4k}(t - 2^k)^2.$$

Therefore, for $-N/3 \leq k \leq -1$,

$$\int_{2^k + 2^k/8 \leq x \leq 2^{k+1} - 2^k/4} \frac{1}{x} \left| \int_{\mathbf{R}} e^{i\lambda\Phi(x-y)} a_N(y) dy \right| dx$$

$$\geq \int_{2^k + 2^k/8 \leq x \leq 2^{k+1} - 2^k/4} \frac{1}{x} \left| \int_{\mathbf{R}} e^{i\lambda(\Phi(2^k) + 2^{2k}(x-y-2^k))} a_N(y) dy \right| dx - C\lambda \cdot 2^{6k}$$

$$= \ln(14/9) \left| \int_{\mathbf{R}} e^{-i\lambda \cdot 2^{2k} \cdot y} a_N(y) dy \right| - C\lambda \cdot 2^{6k}$$

$$= 2\ln(14/9)(2N/3 + 2k + 1)^{-1} - C \cdot 2^{N+6k},$$

where we used (7). Hence we obtain

$$\int_{|x| \geq 2^{-N/3}} \frac{1}{|x|} \left| \int_{\mathbf{R}} e^{i\lambda\Phi(x-y)} a_N(y) dy \right| dx$$

$$\geq 2\ln(14/9) \sum_{-N/3 \leq k \leq -N/4} (2N/3 + 2k + 1)^{-1} - C \sum_{k \leq -N/4} 2^{N+6k}$$

$$\geq c_1 \ln N,$$

where c_1 is some absolute constant. By Proposition 2, we have

$$\|T_\lambda a_N\|_{H^1} \geq \|T_\lambda a_N\|_{L^1} \geq c_1' \ln N \|a_N\|_{H^1},$$

for $\lambda = 2^{N+5}\pi$. Thus we have

$$\sup_{\lambda \in \mathbf{R}} \|T_\lambda\|_{H^1 \to H^1} = \infty. \qquad \square$$

References

1. Carlsson, H., et al., *L^p estimates for maximal functions and Hilbert transforms along flat convex curves in* $\mathbf{R^2}$, Bull. Amer. Math. Soc. **14** (1986), 263–267. MR **87f:**42044
2. Chanillo, S., and Christ, M., *Weak (1,1) bounds for oscillatory singular integrals*, Duke Math. Jour. **55** (1987), 141–155. MR **88h:**42015
3. Hu, Y., and Pan, Y., *Boundedness of oscillatory singular integrals on Hardy spaces*, Ark. Mat. **30** (1993), 311–320. CMP 94:16
4. Nagel, A., Vance, J., Wainger, S., and Weinberg, D., *Hilbert transforms for convex curves*, Duke Math. Jour. **50** (1983), 735–744. MR **85a:**42025
5. Nagel, A., and Wainger, S., *Hilbert transforms associated with plane curves*, Trans. Amer. Math. Soc. **223** (1976), 235–252. MR **54:**10994
6. Pan, Y., *Uniform estimates for oscillatory integral operators*, Jour. Func. Anal. **100** (1991), 207–220. MR **93f:**42034
7. ———, *Boundedness of Oscillatory singular integrals on Hardy spaces: II*, Indiana Univ. Math. Jour **41** (1992), 279–293. CMP 92:11
8. ———, *H^1 boundedness of oscillatory singular integrals with degenerate phase functions*, Math. Proc. Camb. Phil. Soc. **116** (1994), 353–358. MR **95c:**42019
9. Phong, D.H., and Stein, E.M., *Hilbert integrals, singular integrals and Radon transforms I*, Acta Math. **157** (1986), 99–157. MR **88i:**42028a
10. Ricci, F., and Stein, E.M., *Harmonic analysis on nilpotent groups and singular integrals, I*, Jour. Func. Anal. **73** (1987), 179–194. MR **88g:**42023
11. Stein, E.M., *Oscillatory integrals in Fourier analysis*, Beijing Lectures in Harmonic Analysis, Princeton Univ. Press, Princeton, 1986. MR **88g:**42022
12. ———, *Harmonic Analysis: real-variable methods, orthogonality, and oscillatory integrals*, Princeton University Press, Princeton, NJ, 1993. MR **95c:**42002
13. Stein, E.M., and Wainger, S., *Problems in harmonic analysis related to curvature*, Bull. Amer. Math. Soc. **84** (1978), 1239–1295. MR **80k:**42023
14. Zygmund, A., *Trigonometric series*, Cambridge Univ. Press, Cambridge, 1959. MR **21:**6498

Department of Mathematics and Statistics, University of Pittsburgh, Pittsburgh, Pennsylvania 15260

E-mail address: `yibiao@tomato.math.pitt.edu`