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OPERATOR VERSIONS OF THE KANTOROVICH INEQUALITY

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Abstract. The Operator Kantorovich Inequality

$$(R^2 - r^2)\, u^*(a^*a)\, u \;\le\; R^2\, (u^*a^*u)(u^*au)$$

holds for a wide class of operators a on a Hilbert space $\mathcal{H}$ and all operators $u : \mathcal{K} \to \mathcal{H}$ for which $[a]\, u$ is a partial isometry, $[a]$ being the range projection of a.

Introductory

The Kantorovich Inequality

$$4mM\, \langle a^{-1}\xi, \xi \rangle \;\le\; (m+M)^2\, \frac{\|\xi\|^4}{\langle a\,\xi, \xi\rangle},$$

valid for a positive definite matrix with least eigenvalue m and largest eigenvalue M, (and any nonzero vector ξ) first appeared in [K] and has been generalized and applied extensively since.

This inequality may be viewed as a 'reversal' of the special case

$$\langle a\,\xi, \xi\rangle \;\le\; \|a\,\xi\|\, \|\xi\|$$

of the Cauchy-Schwarz inequality, for it is equivalent to the inequality

$$2\sqrt{mM}\, \|a\,\xi\|\, \|\xi\| \;\le\; (m+M)\, \langle a\,\xi, \xi\rangle .$$

The methods of operator and spectral theory allow one to generalize the inequality to a wide class of operators on Hilbert space. If Γ is a nonzero complex number, $R = |\Gamma|$, and $0 \le r \le R$, then

$$(R^2 - r^2)^{\frac{1}{2}}\, \|a\,\xi\|\, \|[a]\,\xi\| \;\le\; R\, \left|\langle a\,\xi, \xi\rangle\right|, \qquad \xi \in \mathcal{H},$$

$[a]$ being the range projection of a, when $a \in \mathcal{C}_{\Gamma, r}$. See below for the definition of $\mathcal{C}_{\Gamma, r}$: it includes all a for which $\sigma(a) \setminus 0$ lies in a disc of radius r with centre Γ, and also the norm-ball of radius r about $\Gamma\, 1_{\mathcal{H}}$.

This generalizes further:

$$(R^2 - r^2)\, u^*(a^*a)\, u \;\le\; R^2\, (u^*a^*u)(u^*au)$$

for $a \in \mathcal{C}_{\Gamma, r}$ and all u for which $[a]\, u$ is a partial isometry.

I am indebted to Dr. Shuangzhe Liu for bringing this topic to my attention.

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Notation & terminology

Recall that when $\mathcal{H}$ is a Hilbert space a *projection* on $\mathcal{H}$ is an *hermitian idempotent* in $L(\mathcal{H})$.

A projection e on $\mathcal{H}$ is a *covering projection* for an operator a on $\mathcal{H}$ if $ea = a$. The *range projection* $[a]$ of a is its smallest covering projection.

An operator $a \in L(\mathcal{H})$ is *positive* if $\langle a\,\xi, \xi \rangle \geq 0$ $(\xi \in \mathcal{H})$: equivalently, if $\sigma(a) \geq 0$, where $\sigma(a)$ is the *spectrum* of a.

Each operator $a \in L(\mathcal{H})$ has a *modulus* $|a| = (a^*a)^{\frac{1}{2}} \geq 0$ in $L(\mathcal{H})$.

Given operators $a \in L(\mathcal{H})$ and $u \in L(\mathcal{K}, \mathcal{H})$ let

$$a^u \;=\; u^*au.$$

Then $|a|^{2u} = u^*(a^*a)\,u$, while $|a^u|^2 = (u^*a^*u)(u^*au)$.

Recall that the operator a is *normal* if $a^*a = a\,a^*$. If e is a covering projection for a normal operator a, then $ae = a$. (*Proof:* verify that $(ae-a)^*(ae-a) = 0$.)

Operators that are invertible on their range. If the operator a is *invertible on its range,* that is, if the operator $\alpha = [a]a[a] = a[a]$ is invertible in $L([a]\mathcal{H})$, then $m \leq |\alpha| \leq M$ on $[a]\,\mathcal{H}$, where $M = \|\alpha\|$, $m = \|\alpha^{-1}\|^{-1}$.

A normal operator a is invertible on its range if and only if *either* $0 \in \rho(a)$ *or* $0 \in \sigma(a)$ and 0 is an isolated point of $\sigma(a)$. If, further, $\sigma(a) \setminus 0$ lies in an open half-plane that does not contain 0, then

$$\sigma(a) \setminus 0 \;\subseteq\; \mathcal{D}$$

for some disc $\mathcal{D}$ not containing 0.

Partially isometric pairs

Let $\mathcal{K}$ be a Hilbert space and let $u \in L(\mathcal{K}, \mathcal{H})$. Recall that the operator u is a *partial isometry* if u^*u is a projection: equivalently, if $u\,u^*$ is a projection.

I shall say that (u, e) is a ***partially isometric pair*** if $e\,u$ (or $u^*\,e$) is a partial isometry: equivalently, if $u^*e\,u$ (or $e\,u\,u^*e$) is a projection. (This condition has already been considered, though only in the finite-dimensional case: see [BP].)

Three sorts of partially isometric pairs are especially to be noticed: *minimal,* $u-$*unitary* and $e-$*identity.*

minimal: For $\xi \in \mathcal{H}$ define

$$u_\xi : \mathbb{C} \to \mathcal{H} : \lambda \to \lambda\,\xi.$$

Then, identifying $\mathbb{C}$ and $L(\mathbb{C})$ canonically,

$$u_\xi^*\,a\,u_\xi \;=\; \langle a\,\xi, \xi \rangle, \qquad a \in L(\mathcal{H}).$$

So $(u_{\xi/\|e\xi\|}, e)$ is a partially isometric pair provided that $e\xi \neq 0$.

$u-$***unitary:*** If $u : \mathcal{K} \to \mathcal{H}$ is unitary then (u, e) is a partially isometric pair for any projection e.

$e-$***identity:*** If $u : \mathcal{K} \to \mathcal{H}$ is a partial isometry then $(u, 1_\mathcal{H})$ is a partially isometric pair.

Fundamental Lemma. *Let $\mathcal{H}, \mathcal{K}$ be Hilbert spaces, let e be a projection on $\mathcal{H}$ and let $u : \mathcal{K} \to \mathcal{H}$ be such that (u, e) is a partially isometric pair:* i.e. *suppose that e^u $(= u^*e\,u)$ is a projection. Then*

$$eue^u \;=\; eu.$$

The *proof* is simple: just multiply out to verify that $(eue^u - eu)^* (eue^u - eu) = 0$.

Corollary. *Let a be an operator on $\mathcal{H}$, let e be a covering projection for a and let $(u,\, e)$ be a partially isometric pair. Then*

$$e^u a^u \;=\; a^u.$$

Further, if a is normal then

$$a^u e^u \;=\; a^u.$$

Fundamental Identity

Let a be an operator on $\mathcal{H}$, let e be a covering projection for a, and let $(u,\, e)$ be a partially isometric pair.

Let Γ be any complex number, let $R = |\Gamma|$, and let r be any real number. Then, using the *Corollary to the Fundamental Lemma,*

$$\begin{aligned}\left|(R^2-r^2)e^u - \overline{\Gamma} a^u\right|^2 \;&=\; \left\{(R^2-r^2)e^u - \Gamma\, a^{*u}\right\}\left\{(R^2-r^2)e^u - \overline{\Gamma}\, a^u\right\} \\ &=\; (R^2-r^2)^2 e^{u2} - (R^2-r^2)\left\{\overline{\Gamma}\, e^u a^u + \Gamma\, a^{*u} e^u\right\} + R^2 |a^u|^2 \\ &=\; (R^2-r^2)^2 e^u - (R^2-r^2)\left\{\overline{\Gamma}\, a^u + \Gamma\, a^{*u}\right\} + R^2 |a^u|^2,\end{aligned}$$

while

$$\begin{aligned}\left\{r^2 e - |a - \Gamma\, e|^2\right\}^u \;&=\; u^* \left\{r^2 e - (a^* - \overline{\Gamma}\, e)(a - \Gamma\, e)\right\} u \\ &=\; -(R^2-r^2)e^u - |a|^{2u} + \overline{\Gamma} a^u + \Gamma a^{*u} :\end{aligned}$$

so

$$\begin{aligned}&R^2 |a^u|^2 - (R^2-r^2)|a|^{2u} \\ &\qquad\equiv\; |(R^2-r^2)e^u - \overline{\Gamma} a^u|^2 + \;(R^2-r^2)\left\{r^2 e - |a - \Gamma\, e|^2\right\}^u,\end{aligned}$$

which is the ***Fundamental Identity.***

The class $\mathcal{C}_{\Gamma,\, r}$

Let Γ be any nonzero complex number, let $R = |\Gamma|$, and suppose that $0 \le r \le R$.

The class $\mathcal{C}_{\Gamma,\, r}$, defined by

$$\mathcal{C}_{\Gamma,\, r} \;\;=\;\; \left\{a \in L(\mathcal{H}) \;:\; |a - \Gamma\,[a]|^2 \le r^2 [a]\right\}$$

(where $[a]$ is the range projection of a), is the natural class of operators for which I can prove an *Operator Kantorovich Inequality.*

The class $\mathcal{C}_{\Gamma,\, r}$ includes

$$\mathcal{N}_{\Gamma,\, r} \;\;=\;\; \Big\{a \in L(\mathcal{H}) :\; a^* a = a\, a^*, \quad \sigma(a) \setminus 0 \;\subseteq\; \mathcal{D}_{\Gamma,\, r}\Big\},$$

where

$$\mathcal{D}_{\Gamma,\, r} \;\;=\;\; \{z \in \mathbb{C} :\; |z - \Gamma| \le r\}\,,$$

and also the norm-ball

$$\mathcal{B}_{\Gamma,\, r} \;\;=\;\; \left\{a \in L(\mathcal{H}) \;:\; \|a - \Gamma\, 1_{\mathcal{H}}\| \le r\right\}.$$

Remark. If a is normal and $a \in \mathcal{C}_{\Gamma,\, r}$ then $|a| \in \mathcal{C}_{|\Gamma|,\, r}$.

Remark. Every normal operator a for which $\sigma(a) \setminus 0$ lies in an open half-plane (not containing 0) belongs to $\mathcal{N}_{\Gamma,\, r}$ for some Γ and r.

Remark. The operator with matrix $\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$ does not belong to $\mathcal{C}_{\Gamma,\, r}$ for any Γ and r.

Positive Operators in $\mathcal{N}_{\Gamma,\, r}$. Suppose that $a \geq 0$ and that a is invertible when restricted to its range: *either* $0 \in \rho(a)$ *or* if $0 \in \sigma(a)$ then 0 is an isolated point of $\sigma(a)$. Let

$$m \;=\; \min\{\sigma(a) \setminus 0\}, \qquad\qquad M \;=\; \max \sigma(a) = \|a\|\,.$$

Then $a \in \mathcal{N}_{\Gamma,\, r}$ with $\Gamma = \frac{m+M}{2}$, $r = \frac{M-m}{2}$.

The inequalities

Let Γ be any nonzero complex number, let $R = |\Gamma|$, and let $0 \leq r \leq R$.

Theorem 1 (Operator Kantorovich Inequality)**.** *Let $\mathcal{H}$ and $\mathcal{K}$ be Hilbert spaces. Suppose that $a \in \mathcal{C}_{\Gamma,\, r}$ and let $[a]$ be the range projection of a. Let $u \in L(\mathcal{K}, \mathcal{H})$ be such that $(u, [a])$ is a partially isometric pair. Then*

$$(R^2 - r^2)\, |a|^{2u} \;\leq\; R^2\, |a^u|^2.$$

Proof. Immediate from the *Fundamental Identity.*

Corollary 1. *Let a be a positive operator on $\mathcal{H}$ which is invertible on its range, let $m = \min\{\sigma(a) \setminus 0\}$, $M = \max \sigma(a) = \|a\|$. Let $u \in L(\mathcal{K}, \mathcal{H})$ be such that $(u, [a])$ is a partially isometric pair. Then*

$$2\sqrt{mM}\, (a^{2u})^{\frac{1}{2}} \;\leq\; (m+M)\, a^u.$$

This follows from Theorem 1 and from the uniqueness and monotonicity of square roots.

Corollary 2 (Scalar Version — Minimal Partially Isometric Pairs)**.** *Let $a \in \mathcal{C}_{\Gamma,\, r}$, and let $[a]$ be its range projection. Then*

$$(R^2 - r^2)^{\frac{1}{2}}\; \|a\,\xi\|\; \|[a]\,\xi\| \;\leq\; R \left| \left\langle a\,\xi,\, \xi \right\rangle \right|, \qquad \xi \in \mathcal{H}.$$

If a is positive with $\sigma(a) \setminus 0 \subseteq\; [m,\, M] \quad (0 < m \leq M)$, then

$$2\sqrt{mM}\; \|a\,\xi\|\; \|[a]\,\xi\| \;\leq\; (m+M)\; \left\langle a\,\xi,\, \xi \right\rangle, \qquad \xi \in \mathcal{H}.$$

Proof. There is nothing to prove if $[a]\xi = 0$: otherwise put $u = u_{\xi/\|[a]\xi\|}$ and take square roots. The second assertion is a direct consequence of the first.

Remark. The second assertion may be proved 'in one line':

$$\begin{aligned} &(m+M)^2\, \left\langle a\,\xi,\, \xi \right\rangle^2 \;-\; 4mM\; \|a\,\xi\|^2\; \|[a]\,\xi\|^2 \\ &\quad=\; \Big\{ 2mM\; \|[a]\,\xi\|^2 - (m+M) \left\langle a\,\xi,\, \xi \right\rangle \Big\}^2 \\ &\qquad+\; 4mM \Big\langle (M-a)(a-m)\,[a]\,\xi,\, [a]\,\xi \Big\rangle\, \|[a]\,\xi\|^2 \\ &\quad\geq\; 0. \end{aligned}$$

Remark. The example $a = \operatorname{diag}\{1, -1\}$ shows that the hypothesis on the spectrum is essential. The example $a = \operatorname{diag}\{m, M\}$, $\quad \xi = (M^{\frac{1}{2}}, m^{\frac{1}{2}})^*$ shows that the ratio $\frac{m+M}{2\sqrt{mM}}$ may be attained so cannot be bettered.

Corollary 2 can be 'polarized':

Corollary 3. *Let $a \in \mathcal{C}_{\Gamma,\,r}$ and let $[a]$ be its range projection. Then*

$$(R^2 - r^2)\ \left|\left\langle a\,\xi,\, a\,\eta\right\rangle\ \left\langle [a]\,\xi,\, [a]\,\eta\right\rangle\right|\ \le\ R^2\ \left|\left\langle a\,\xi,\, \xi\right\rangle\ \left\langle a\,\eta,\, \eta\right\rangle\right|, \qquad \xi,\, \eta \in \mathcal{H}.$$

The *proof:*

$$\begin{aligned}(R^2 - r^2)\ \left|\left\langle a\,\xi,\, a\,\eta\right\rangle\ \left\langle [a]\,\xi,\, [a]\,\eta\right\rangle\right|\ &\le\ (R^2 - r^2)\ \|a\,\xi\|\ \|a\,\eta\|\ \|[a]\,\xi\|\ \|[a]\,\eta\|\\ &=\ (R^2 - r^2)^{\frac{1}{2}}\ \|a\,\xi\|\ \|[a]\,\xi\|\ (R^2 - r^2)^{\frac{1}{2}}\ \|a\,\eta\|\ \|[a]\,\eta\|\end{aligned}$$

which, by Corollary 2,

$$\begin{aligned}&\le\ R\ \left|\left\langle a\,\xi,\, \xi\right\rangle\right| R\ \left|\left\langle a\,\eta,\, \eta\right\rangle\right|\\ &=\ R^2\ \left|\left\langle a\,\xi,\, \xi\right\rangle\ \left\langle a\,\eta,\, \eta\right\rangle\right|.\end{aligned}$$

The next corollary provides an improvement of both the result and the proof of Strang [S].

Corollary 4. *Suppose that a is an operator on the Hilbert space $\mathcal{H}$ and that a is invertible on its range:* i.e. *the operator $\alpha = eae$ is invertible in $L(e\mathcal{H})$, where e is the range projection of a. Let $M\ =\ \|\alpha\|,\quad m\ =\ \left\|\alpha^{-1}\right\|^{-1}$. Then*

$$\begin{aligned}4mM\ \left|\left\langle ae\xi,\, ae\eta\right\rangle\ \left\langle e\xi,\, e\eta\right\rangle\right|\ &\le\ (m+M)^2\ \left\langle |\alpha|\,\xi,\, \xi\right\rangle\ \left\langle |\alpha|\,\eta,\, \eta\right\rangle,\\ 4mM\ \left|\left\langle ae\xi,\, e\eta\right\rangle\ \left\langle e\xi,\, a^{-1}\,e\eta\right\rangle\right|\ &\le\ (m+M)^2\ \|e\xi\|^2\ \|e\eta\|^2, \qquad \xi,\, \eta \in \mathcal{H}.\end{aligned}$$

Proof. I abuse notation conventionally and write $a^{-1}\,e\eta$ in place of $\alpha^{-1}\,e\eta$.

As remarked above, $m \le |\alpha| \le M$ on $e\,\mathcal{H}$. Corollary 3 applied to the operator α and vectors $e\xi$, $e\eta$ gives the first conclusion.

Now given $\xi,\, \eta \in \mathcal{H}$ there are (unique) $\theta,\, \phi \in e\mathcal{H}$ such that $e\xi = |\alpha|^{\frac{1}{2}}\,\theta,\ e\eta = \alpha|\alpha|^{-\frac{1}{2}}\,\phi$. Then

$$\begin{aligned}\left\langle ae\xi,\, e\eta\right\rangle\ &=\ \left\langle |\alpha|\,\theta,\, |\alpha|\,\phi\right\rangle,\\ \left\langle e\xi,\, a^{-1}\,e\eta\right\rangle\ &=\ \left\langle \theta,\, \phi\right\rangle,\\ \|e\xi\|^2\ &=\ \left\langle |\alpha|\,\theta,\, \theta\right\rangle,\\ \|e\eta\|^2\ &=\ \left\langle |\alpha|\,\phi,\, \phi\right\rangle.\end{aligned}$$

Thus

$$4mM\ \left|\left\langle ae\xi,\, \eta\right\rangle\ \left\langle e\xi,\, a^{-1}e\eta\right\rangle\right|\ =\ 4mM\ \left|\left\langle |\alpha|\,\theta,\, |\alpha|\,\phi\right\rangle\ \left\langle \theta,\, \phi\right\rangle\right|$$

which, by the first part of this corollary,

$$\begin{aligned}&\le\ (m+M)^2\ \left\langle |\alpha|\,\theta,\, \theta\right\rangle\ \left\langle |\alpha|\,\phi,\, \phi\right\rangle\\ &=\ (m+M)^2\ \|e\,\xi\|^2\ \|e\,\eta\|^2.\end{aligned}$$

This immediately specialises to yield

Corollary 5 (Original Kantorovich Inequality)**.** *If a is positive and invertible, with $\sigma(a) \subseteq\ [m,\, M]\quad (0 < m \le M)$, then*

$$4mM\ \left\langle a^{-1}\xi,\, \xi\right\rangle\ \le\ (m+M)^2\ \frac{\|\xi\|^4}{\left\langle a\,\xi,\, \xi\right\rangle}, \qquad \xi \in \mathcal{H}.$$

Remark. This may also be shown 'in one line':

$$\begin{aligned}
&(m+M)^2\,\|\xi\|^4 \;-\; 4mM\,\left\langle a\,\xi,\,\xi\right\rangle\,\left\langle a^{-1}\xi,\,\xi\right\rangle \\
&\quad=\;\Big\{2mM\,\left\langle a^{-1}\xi,\,\xi\right\rangle-(m+M)\,\|\xi\|^2\Big\}^2 \\
&\qquad+\;4mM\,\Big\langle (M-a)(1-m\,a^{-1})\,\xi,\,\xi\Big\rangle\,\left\langle a^{-1}\xi,\,\xi\right\rangle \\
&\quad\geq\;0.
\end{aligned}$$

'Duplex' inequalities

Suppose that a and b are operators on $\mathcal{H}$ and that e is a common covering projection for a and b : *i.e.* $ea = a$, $eb = b$. Suppose further that $(u,\,e)$ is a partially isometric pair.

Let Γ be any complex number, let $R = |\Gamma|$, and let r be any real number. Then

$$\begin{aligned}
&\left|2(R^2-r^2)e^u-\overline{\Gamma}(a+b)^u\right|^2 \\
&\quad=\;\big\{2(R^2-r^2)e^u-\Gamma(a+b)^{*u}\big\}\big\{2(R^2-r^2)e^u-\overline{\Gamma}(a+b)^u\big\} \\
&\quad=\;4\,(R^2-r^2)^2e^u+R^2|(a+b)^u|^2-2\,(R^2-r^2)\big\{\overline{\Gamma}\,(a+b)^u+\Gamma\,(a+b)^{*u}\big\}
\end{aligned}$$

while

$$\begin{aligned}
&\big\{|a-b|^2+2r^2e-|a-\Gamma e|^2-|b-\Gamma e|^2\big\}^u \\
&\qquad\qquad=\;-(a^*b+b^*a)^u-2(R^2-r^2)e^u+\overline{\Gamma}(a+b)^u+\Gamma(a+b)^{*u}\;:
\end{aligned}$$

so

$$\begin{aligned}
R^2|(a+b)^u|^2&-2(R^2-r^2)(a^*b+b^*a)^u \\
&\equiv\;|2(R^2-r^2)e^u-\overline{\Gamma}(a+b)^u|^2 \\
&\qquad+\;2(R^2-r^2)\,\big\{|a-b|^2+2r^2e-|a-\Gamma e|^2-|b-\Gamma e|^2\big\}^u,
\end{aligned}$$

which is the ***Duplex Fundamental Identity.*** The next theorem follows from this.

Theorem 2 (Duplex Kantorovich Inequality)**.** *Suppose that a and b are operators on $\mathcal{H}$ with common range projection e and that both a and b belong to the class $\mathcal{C}_{\Gamma,\,r}$. Suppose also that $(u,\,e)$ is a partially isometric pair. Then*

$$2(R^2-r^2)(a^*b+b^*a)^u\;\leq\;R^2|(a+b)^u|^2.$$

When $a = b$ one recovers

$$(R^2-r^2)\,|a|^{2u}\;\leq\;R^2\,|a^u|^2.$$

Theorem 3 (Positive Duplex Case)**.** *Suppose that a and b are both positive operators on $\mathcal{H}$, that they commute, have common range projection e, and are invertible on their range. Let*

$$\begin{aligned}
m_1&=\min\{\sigma(a)\setminus 0\}, &\qquad M_1&=\max\sigma(a),\\
m_2&=\min\{\sigma(b)\setminus 0\}, &\qquad M_2&=\max\sigma(b).
\end{aligned}$$

Then

$$4(m_1M_2+m_2M_1)\,(ab)^u\;\leq\;|(m_2a+M_1b)^u|^2\;+\;|(M_2a+m_1b)^u|^2$$

for

$$\begin{aligned}|(m_2a+M_1b)^u|^2 \;+\; |(M_2a+m_1b)^u|^2 \;-\; 4(m_1M_2+m_2M_1)(ab)^u \\ = \; |2m_2M_1e^u-(m_2a+M_1b)^u|^2 + \; |2m_1M_2e^u-(M_2a+m_1b)^u|^2 \\ + \; 4m_2M_1\big\{e(M_1-a)(b-m_2)e\big\}^u \;+\; 4m_1M_2\big\{e(M_2-b)(a-m_1)e\big\}^u.\end{aligned}$$

This may be compared with the *Generalized Pólya-Szegö Inequality* of Greub & Rheinboldt [GR], which the next theorem extends.

Theorem 4. *Suppose that a and b are commuting normal operators, with common range projection e, and are invertible on their range. Let*

$$m_1=\|(eae)^{-1}\|^{-1}, \qquad M_1=\|eae\|,$$
$$m_2=\|(ebe)^{-1}\|^{-1}, \qquad M_2=\|ebe\|.$$

Then

$$\begin{aligned}&4m_1m_2M_1M_2\,\big|\big\langle a\,\xi,\,a\,\eta\big\rangle\;\big\langle b\,\xi,\,b\,\eta\big\rangle\big| \\ &\qquad\le\;(m_1m_2+M_1M_2)^2\;\big\langle |a|\,\xi,\,|b|\,\xi\big\rangle\;\big\langle |a|\,\eta,\,|b|\,\eta\big\rangle\,, \qquad \xi,\eta\in\mathcal{H}.\end{aligned}$$

If $(eae)(ebe)^{-1}\ge 0$, if, in particular, $a\ge 0,\; b\ge 0$, then

$$\begin{aligned}&4m_1m_2M_1M_2\,\big|\big\langle a\,\xi,\,a\,\eta\big\rangle\;\big\langle b\,\xi,\,b\,\eta\big\rangle\big| \\ &\qquad\le\;(m_1m_2+M_1M_2)^2\;\big\langle a\,\xi,\,b\,\xi\big\rangle\;\big\langle a\,\eta,\,b\,\eta\big\rangle\,, \qquad \xi,\eta\in\mathcal{H}.\end{aligned}$$

Proof. Note that $\big\|(eae)(ebe)^{-1}\big\|\le\frac{M_1}{m_2}$, while $\big\|(ebe)(eae)^{-1}\big\|^{-1}\ge\frac{m_1}{M_2}$; apply Corollary 4 to the operator $(eae)(ebe)^{-1}$ and the vectors $b\,\xi$ and $b\,\eta$.

References

[BP] JK Baksalary & S Puntanen, *Generalized matrix versions of the Cauchy-Schwarz and Kantorovich inequalities*, Aequationes Mathematicae **41** (1991), 103-110. MR **91k:**15038

[GR] W Greub & W Rheinboldt, *On a generalization of an inequality of L.V. Kantorovich*, Proc American Math Soc **10** (1959), 407-415. MR **21:**3774

[K] L V Kantorovich, *Functional analysis and applied mathematics* (Russian), Uspekhi Mat Nauk (NS) **3** (1948), 89-185. MR **10:**380a

[S] W G Strang, *On the Kantorovich Inequality*, Proc American Math Soc **11** (1959) 468. MR **22:**2904

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