

SOME REMARKS ON THE OPERATOR OF FOIAS AND WILLIAMS

SRDJAN PETROVIĆ

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ABSTRACT. In this paper we study the Foias-Williams operator

$$T(H_g) = \begin{pmatrix} S^* & H_g \\ 0 & S \end{pmatrix}$$

where $g \in L^\infty$, and H_g is a Hankel operator with symbol g . We exhibit a relationship between the similarity of $T(H_g)$ to a contraction and the rate of decay of $\{|g_n|\}_{n=0}^\infty$, the absolute values of the Fourier coefficients of the symbol g .

Let \mathcal{H} denote a complex Hilbert space and let $\mathcal{L}(\mathcal{H})$ denote the algebra of all bounded linear operators on \mathcal{H} . Recall that an operator T in $\mathcal{L}(\mathcal{H})$ is said to be *polynomially bounded* if there exists an $M \geq 1$ such that

$$\|p(T)\| \leq M \sup\{|p(\zeta)| : |\zeta| = 1\}$$

for all polynomials p . We denote the class of all polynomially bounded operators by (PB). Also, an operator T is similar to a contraction (notation: $T \in (\text{SC})$) if there exists a bounded invertible operator L such that $\|LTL^{-1}\| \leq 1$. Halmos [4] raised the question whether every polynomially bounded operator is similar to a contraction; this question is still open.

While there is a number of results dealing with sufficient conditions for a polynomially bounded operator to be similar to a contraction (cf. [9], [6], [5]), there are very few publications dedicated to the search for a counterexample. In [3] (see also [2]) Foias and Williams have studied the operators of the form

$$T(X) = \begin{pmatrix} S^* & X \\ 0 & S \end{pmatrix}$$

acting on $H^2 \oplus H^2$, the direct orthogonal sum of two copies of the Hardy space H^2 (to be defined below). Here S is a forward unilateral shift on H^2 and $X \in \mathcal{L}(H^2)$. In particular, they conjectured that there exists a Hankel operator H_g with symbol g (to be defined below) such that the operator $T(H_g) \in (\text{PB}) \setminus (\text{SC})$.

In this paper we continue this line of investigation and we show that the membership in the aforementioned classes depends on the rate at which the sequence $\{g_n\}_{n \in \mathbb{N}_0}$ of the Fourier coefficients of g tends to 0.

Before stating the main results, however, we briefly introduce some notation and terminology. As usual \mathbb{N} [resp., \mathbb{N}_0] denotes the set of positive [resp., non-negative]

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integers, \mathbb{C} is the complex plane and \mathbb{T} is the unit circle in \mathbb{C} . L^∞ is the algebra of all bounded Lebesgue-measurable functions on \mathbb{T} , and H^∞ is its subalgebra consisting of those functions whose negative Fourier coefficients vanish. Similarly, H^2 is the space of square-integrable functions on \mathbb{T} whose negative Fourier coefficients vanish. Finally, ℓ^2 is the Hilbert space of all sequences $(x_n)_{n \in \mathbb{N}_0}$ such that $\sum_{n \in \mathbb{N}_0} |x_n|^2 < \infty$

and $\|(x_n)\|_{\ell^2} = \left\{ \sum_{n \in \mathbb{N}_0} |x_n|^2 \right\}^{1/2}$. Furthermore, $z^n = e^{int}$, $n \in \mathbb{N}_0$, is the standard orthonormal basis of H^2 , e_n , $n \in \mathbb{N}_0$, is the standard basis of ℓ^2 (e_n is a sequence whose only non-zero term is in the n th position), and there exists a Hilbert space isomorphism $V : H^2 \rightarrow \ell^2$ such that $Vz^n = e_n$, $n \in \mathbb{N}_0$. In what follows we shall identify H^2 with ℓ^2 (and z^n with e_n) without further comment. If $g \in L^\infty$, H_g is the Hankel operator with symbol g , i.e., H_g is a bounded linear operator on H^2 satisfying $H_g(z^n) = S^{*n}g$. In this situation, using the aforementioned identification between H^2 and ℓ^2 , H_g is represented by the matrix $(g_{i+j})_{i,j=0}^\infty$, where g_n denotes the n th Fourier coefficient of g .

Lemma 1. *Suppose $g \in H^\infty$ and the series $\sum_{n=1}^\infty n^2|g_n|^2$ does not converge. Then $T(H_g)$ is not similar to a contraction. In particular, this happens when $|g_n| = (n + 1)^{-3/2}$.*

Proof. By [2, Theorem 5.3], $T(X) \in (\text{SC})$ if and only if $\exists f \in H^2$ such that the operator W defined on polynomials by

$$Wz^n = S^*(A_{n+1} \cdot 1 - S^{*n}f), \quad n \in \mathbb{N}_0,$$

extends to a bounded linear operator on H^2 . Here

$$A_n = \sum_{j=0}^{n-1} S^{*(n-1-j)}XS^j, \quad n \in \mathbb{N}.$$

In particular, when $X = H_g$, the Hankel operator with symbol $g \in L^\infty(\mathbb{T})$,

$$A_n = \sum_{j=0}^{n-1} S^{*n-1-j}H_gS^j = \sum_{j=0}^{n-1} S^{*n-1}H_g = nS^{*n-1}H_g,$$

and therefore

$$Wz^n = S^*((n + 1)S^{*n}H_g \cdot 1 - S^{*n}f) = (n + 1)S^{*n+1}H_g \cdot 1 - S^{*n+1}f, \quad n \in \mathbb{N}_0.$$

Since $H_g1 = g$ we have that $Wz^n = (n + 1)S^{*n+1}g - S^{*n+1}f$. Clearly, such an operator W , in the basis $\{z^n\}_{n \in \mathbb{N}_0}$ is represented by a matrix

$$((Wz^r, z^s))_{r,s=0}^\infty$$

and it is easy to see that

$$\langle Wz^r, z^s \rangle = (r + 1)\langle g, z^{r+s+1} \rangle - \langle f, z^{r+s+1} \rangle = (r + 1)g_{r+s+1} - f_{r+s+1}.$$

In other words, W is represented by a matrix

$$(1) \quad A = ((r + 1)g_{r+s+1} - f_{r+s+1})_{r,s=0}^\infty,$$

and for W to be bounded on H^2 it is necessary that the columns of A be square summable, i.e.,

$$\sum_r |(r + 1)g_{r+s+1} - f_{r+s+1}|^2 < \infty, \quad \forall s \in \mathbb{N}_0.$$

In particular we have that $\sum_r |(r + 1)g_r - f_r|^2 < \infty$, which in view of $f \in H^2$ implies that $\sum_r r^2 |g_r|^2 < \infty$, and the lemma is proved. \square

The main result of this paper is that $-\frac{3}{2}$ is indeed a limiting point. More precisely, we have the following

Theorem 2. *Suppose that $\alpha > \frac{3}{2}$. Then there exists a function $g \in H^\infty$ such that $|g_n| = \frac{1}{(n + 1)^\alpha}$, $n \in \mathbb{N}_0$ and such that, with $f = 0$, the matrix (1) is bounded on ℓ^2 .*

Before we can prove Theorem 2 we need to recall a few results. First, one knows that there exists a sequence $\{\epsilon_n\}_{n \in \mathbb{N}_0} \subset \{-1, 1\}$ and a constant C such that

$$(2) \quad \sup_{z \in \mathbb{T}} \left| \sum_{n=0}^N \epsilon_n z^n \right| \leq C\sqrt{N}, \quad \forall N \in \mathbb{N}.$$

Since the numbers ϵ_n were independently introduced by Shapiro [8] and Rudin [7], they are generally referred to as the Rudin-Shapiro coefficients.

Next, we recall that for two matrices $A = (a_{ij})$ and $B = (b_{ij})$ the Schur product of A and B is defined as $A * B = (a_{ij}b_{ij})$. A matrix M is said to be a Schur multiplier if $M * X$ defines a bounded operator on ℓ^2 for every $X \in \mathcal{L}(\ell^2)$. The following useful test comes from [1]:

Theorem 3. *Let M be a matrix with $\lim_k m_{jk} = 0 = \lim_j m_{jk}$, and suppose that*

$$\sum_{j,k=1}^\infty |m_{j,k} - m_{j,k+1} - m_{j+1,k} + m_{j+1,k+1}| < \infty.$$

Then M is a Schur multiplier.

Proof of Theorem 2. Given $\alpha > \frac{3}{2}$, we define $g_n = \frac{\epsilon_n}{(n + 1)^\alpha}$, $n \in \mathbb{N}_0$, where ϵ_n denotes the n th Rudin-Shapiro coefficient. In this situation, with $f = 0$, the matrix (1) becomes

$$A = \left((r + 1) \frac{\epsilon_{r+s}}{(r + s + 1)^\alpha} \right)_{r,s=0}^\infty.$$

Since $\alpha > \frac{3}{2}$, $\exists \beta > 0$ such that $\alpha = 2\beta + \frac{3}{2}$. Using the Schur product we can rewrite $A = M * B$ where

$$M = \left(\frac{r + 1}{(r + s + 1)^{1+\beta}} \right), \quad B = \left(\frac{\epsilon_{r+s}}{(r + s + 1)^{\frac{1}{2}+\beta}} \right).$$

First we show that B defines a bounded linear operator on ℓ^2 . We notice that B is the Hankel operator corresponding to the function $h(z) = \sum_{k=0}^{\infty} \frac{\epsilon_k}{(k+1)^{\frac{1}{2}+\beta}} z^k$.

Therefore it suffices to show that $h \in H^\infty$ because in that case $\|B\| \leq \|h\|_\infty$.

An easy application of the Abel summation by parts allows us to rewrite

$$h(z) = \sum_{k=0}^{\infty} \left(\sum_{j=0}^k \epsilon_j z^j \right) \left[\frac{1}{(k+1)^{\frac{1}{2}+\beta}} - \frac{1}{(k+2)^{\frac{1}{2}+\beta}} \right]$$

and therefore

$$\|h\|_\infty \leq \sup_{z \in \mathbb{T}} \sum_{k=0}^{\infty} \left| \sum_{j=0}^k \epsilon_j z^j \right| \frac{(k+2)^{\frac{1}{2}+\beta} - (k+1)^{\frac{1}{2}+\beta}}{(k+1)^{\frac{1}{2}+\beta} (k+2)^{\frac{1}{2}+\beta}}.$$

By the mean value theorem

$$(k+2)^{\frac{1}{2}+\beta} - (k+1)^{\frac{1}{2}+\beta} = \left(\frac{1}{2} + \beta \right) (k+1+\theta)^{\frac{1}{2}+\beta-1}$$

for some $0 < \theta < 1$. Thus, employing (2),

$$\|h\|_\infty \leq \sum_{k=0}^{\infty} C\sqrt{k} \left(\frac{1}{2} + \beta \right) \frac{(k+1+\theta)^{\beta-\frac{1}{2}}}{(k+1)^{\frac{1}{2}+\beta} (k+2)^{\frac{1}{2}+\beta}}$$

and it is easy to see that this series converges, from which it follows that B is a bounded operator.

Finally, we show that M is a Schur multiplier. We apply Theorem 3 with $m_{jk} = \frac{j+1}{(j+k+1)^{1+\beta}}$. Clearly, both limits are 0. Therefore, it remains to show the convergence of the double series

$$\sum_{j,k=1}^{\infty} \left| \frac{j+1}{(j+k+1)^{1+\beta}} - \frac{j+1}{(j+k+2)^{1+\beta}} - \frac{j+2}{(j+k+2)^{1+\beta}} + \frac{j+2}{(j+k+3)^{1+\beta}} \right|.$$

One knows that if a function $f(x, y)$ has continuous partial derivatives of the second order, then

$$(3) \quad f(x, y) - f(x, y+1) - f(x+1, y) + f(x+1, y+1) = \frac{\partial^2 f}{\partial x \partial y}(x+\theta, y+\eta),$$

for some $0 < \theta, \eta < 1$. Using (3) with $f(x, y) = \frac{x+1}{(x+y+1)^{1+\beta}}$, we obtain that the series above can be written as

$$(\beta+1) \sum_{j,k=1}^{\infty} \left| \frac{(\beta+1)(j+\theta+1) - (k+\eta)}{(j+k+\theta+\eta+1)^{\beta+3}} \right|.$$

Since this double series is obviously convergent, the theorem is proved. \square

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DEPARTMENT OF MATHEMATICS, INDIANA UNIVERSITY, BLOOMINGTON, INDIANA 47405
E-mail address: petrovic@iu-math.math.indiana.edu