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GEOMETRIC INDICES AND THE ALEXANDER POLYNOMIAL OF A KNOT

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Abstract. It is well-known that any Laurent polynomial $\Delta(t)$ satisfying $\Delta(t) \doteq \Delta(t^{-1})$ and $\Delta(1) = \pm 1$ is the Alexander polynomial of a knot in S^3. We show that $\Delta(t)$ can be realized by a knot which has the following properties simultaneously: (i) tunnel number 1; (ii) bridge index 3; and (iii) unknotting number 1.

1. Introduction

In this paper, we study some geometric indices of a knot in the 3-sphere S^3: the tunnel number, the unknotting number, the bridge index, and the Alexander polynomial. The tunnel number of a knot K, $\tau(K)$, is the minimum number of mutually disjoint arcs properly embedded in the exterior of K such that the complementary space of $K \cup \{\text{arcs}\}$ is a handlebody. The unknotting number of K, $u(K)$, is the minimum number of exchanges of crossings required to deform K into a trivial knot over all knot diagrams representing K. The bridge index of K, $b(K)$, is the minimum number of n such that (S^3, K) is a tangle sum $(B_1^3, t_1) \cup (B_2^3, t_2)$, where (B_i^3, t_i) is a trivial n-string tangle, $i = 1, 2$. We denote by $\Delta_K(t)$ the Alexander polynomial of K.

Theorem 1. *For any Laurent polynomial $A(t) \in Z[t, t^{-1}]$ with $A(1) = \pm 1$ and $A(t) \doteq A(t^{-1})$, where $\doteq$ means "equal up to units", there exists a knot K in S^3 which satisfies the following conditions simultaneously:*

(i) $\Delta_K(t) = A(t)$,
(ii) $\tau(K) = 1$,
(iii) $b(K) = 3$, *and*
(iv) $u(K) = 1$.

Theorem 2. *There exists a set of mutually inequivalent knots $\{K_i\}_{i \in Z}$ such that each K_i satisfies the following conditions simulaneously:*

(i) $\Delta_K(t) = 1$,
(ii) $\tau(K) = 1$,
(iii) $b(K) = 3$, *and*
(iv) $u(K) = 1$.

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Note that a tunnel number one knot is strongly invertible [1]. For any Laurent polynomial $A(t)$ with $A(1) = \pm 1$ and $A(t) \doteq A(t^{-1})$, Kondo [8] and Sakai [17] constructed an unknotting number one knot whose Alexander polynomial is equal to $A(t)$. And then Sakai [18] showed that his knots are strongly invertible.

Let F_g be a genus g surface standardly embedded in S^3. Any knot can be embedded in F_g for some g. A torus knot is a knot which is embedded in F_1. According to the observation of C. Morin and M. Saito; cf. [13, p.138], a tunnel number one knot can be embedded in either F_1 or F_2.

From Theorem 1, we get the following result.

Corollary. *For any Laurent polynomial $A(t)$ with $A(1) = \pm 1$ and $A(t) \doteq A(t^{-1})$, there exists a knot embedded in F_2 whose Alexander polynomial is equal to $A(t)$.*

The Alexander polynomial of a torus knot has some restricted form. This is also true for the Alexander polynomial of a 2-bridge knot (cf. Hartley [5]). Sakuma proposed the following conjecture.

Conjecture. *For any Laurent polynomial $A(t)$ with $A(1) = \pm 1$ and $A(t) \doteq A(t^{-1})$, there exists a tunnel number one knot* (*resp. a* 3*-bridge knot*) *whose Alexander polynomial is equal to $A(t)$.*

Theorem 1 implies that this conjecture is true. The proofs are given in Sections 2 and 3; they are obtained by a small alteration of Sakai's construction in [17].

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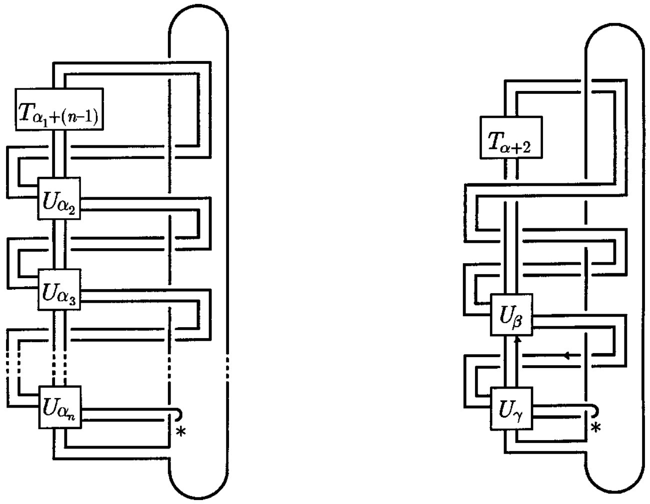
2. Geometric indices

To prove Theorems 1 and 2, we consider two classes of knots $K(\alpha_1, \cdots, \alpha_n)$ and $J(\alpha, \beta, \gamma)$ as illustrated in Figure 1.



a b

Figure 1

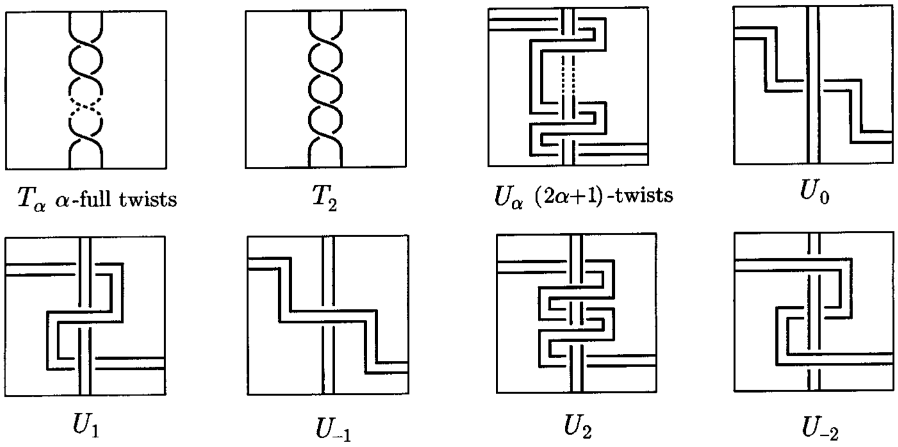


Figure 2

Here T_α is a 2-string tangle with left hand α-full twists, and U_α is a 4-string tangle with left hand $(2\alpha+1)$-half twists of two bands; see Figure 2. Note that "the band parts" of U_α are untwisted.

Then we easily see the upper bounds of the geometric indices.

Lemma 1. *The unknotting numbers of* $K(\alpha_1, \cdots, \alpha_n)$ *and* $J(\alpha, \beta, \gamma)$ *are less than or equal to one.*

Proof. Exchange the crossings $*$ in Figures 1a and 1b, and we get a trivial knot. □

Lemma 2. *The tunnel numbers of* $K(\alpha_1, \cdots, \alpha_n)$ *and* $J(\alpha, \beta, \gamma)$ *are less than or equal to one.*

Proof. Let $K = K(\alpha_1, \cdots, \alpha_n)$. For simplicity, we prove the lemma only in the case of $n = 4$. The proof in the general case is similar. We add the arc τ in $E(K)$ as illustrated in Figure 3a, and we have the deformations indicated in Figures 3a–3d. Then τ_2 in Figure 3d is an unknotting tunnel of the 2-bridge knot; see [1]. Hence $\tau(K) \leq 1$.

For $J(\alpha, \beta, \gamma)$, the unknotting tunnel τ and the deformation are given in Figures 4a–4c. □

To state Lemma 3, we recall the following results.

Proposition 1 ([12])**.** *Let* K *be a knot, and* $\Sigma_2(K)$ *the two-fold branched covering space of* S^3 *branched over* K*. If the unknotting number of* K *is one, then there exists a strongly invertible knot* K' *such that* $\Sigma_2(K)$ *can be obtained by the Dehn surgery of type* $p/2$ *on* K'*, where* p *is an odd integer,*

Proposition 2 (Cyclic Surgery Theorem [3])**.** *Let* K *be a knot which is not a torus knot, and* $K(r)$ *the* 3*-manifold obtained by the Dehn surgery of type* r *on* K $(r \in Q)$*. If the fundamental group* $\pi_1(K(r))$ *is cyclic, then* r *is an integer.*

We denote by $C(\alpha_1, \cdots, \alpha_n)$ $(\alpha_i \in Z)$ the 2-bridge knot as shown in Figure 5; see [2].

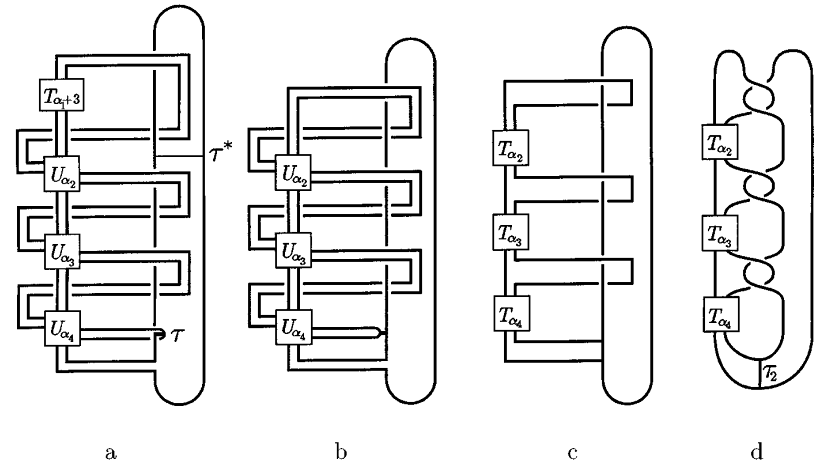


FIGURE 3

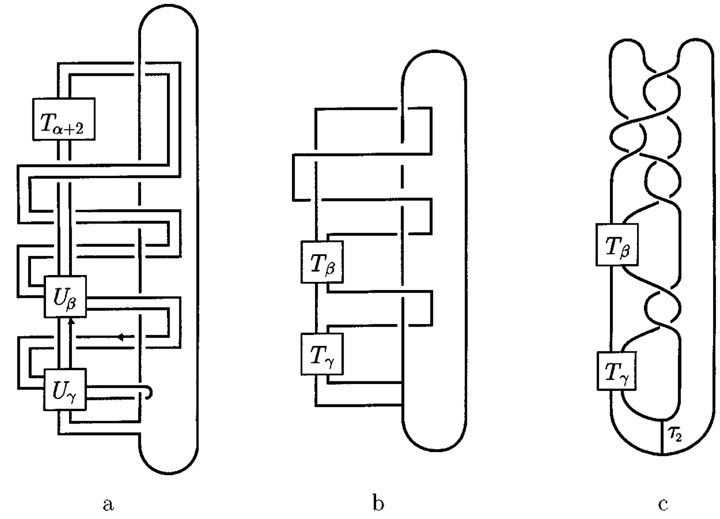


FIGURE 4

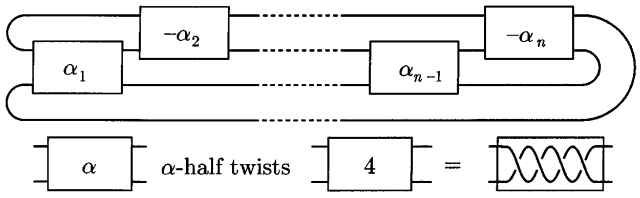


FIGURE 5

Proposition 3 ([6])**.** *We have the transformations of* 2*-bridge knots as follows:*

$$
\begin{aligned}
(1)\qquad & C(x_1,\cdots,x_n,a,y,y_1,\cdots,y_m)\\
\cong\ & C(x_1,\cdots,x_n+\epsilon,||a||,(-1)^a(y+\epsilon),(-1)^a y_1,\cdots,(-1)^a y_m),
\end{aligned}
$$

$$C(x_1,\cdots,x_n,a)\cong C(x_1,\cdots,x_n+\epsilon,||a||),$$

where $\epsilon=a/|a|$ *and* $||a||=(\underbrace{-2\epsilon,2\epsilon,\cdots,(-1)^{|a|-1}2\epsilon}_{|a|-1})$.

Lemma 3. *Let* K *be one of* $K(\alpha_1,\cdots,\alpha_n)$ $(n\geq 3$ *and* $\alpha_n\neq 0)$ *and* $J(\alpha,\beta,\gamma)$ $(\beta\neq 0$ *or* $\gamma\neq 0)$*. Then* $b(K)=3$*.*

Proof. Case 1. Let $K=K(\alpha_1,\cdots,\alpha_n)$ ($n\geq 3$ and $\alpha_n\neq 0$). It is easy to see that $b(K)\leq 3$. We show that $b(K)\geq 3$. If K is a 2-bridge knot, then the two-fold branched covering space of S^3 branched over K, denoted by $\Sigma_2(K)$, is a lens space. So $\pi_1(\Sigma_2(K))$ is cyclic.

From Proposition 1 and the construction given in [12], we have that $\Sigma_2(K)$ is a 3-manifold obtained by Dehn surgery of type $p/2$ on the 2-bridge knot $C=C(-1,-4a_2,-1,-4a_3,-1,\cdots,-4a_n)$, where p is an odd integer; see Figure 6 in Section 3. By Proposition 2, it is sufficient to prove that C is not a torus knot. If C is a torus knot, then $C\cong\pm C(\underbrace{2,-2,\cdots,2,-2}_{|m|-1})$ which is a torus knot of type $(2,m)$. Using Proposition 3, we deform C as follows:

(2)

$$
\begin{aligned}
& C(-1,-4a_2,-1,-4a_3,-1,-4a_4,-1,\cdots,-4a_n)\\
\cong\ & C(-1+\epsilon_2,||-4a_2||,-1+\epsilon_2,-4a_3,-1,-4a_4,-1,\cdots,-4a_n)\\
\cong\ & \begin{cases}
C(0,\underbrace{-2,2,\cdots,-2}_{|4a_2|-1},0,-4a_3,-1,-4a_4,-1,\cdots,-4a_n), & \text{if } \epsilon_2=1;\\
C(-2,\underbrace{2,-2,\cdots,2}_{|4a_2|-1},-2,-4a_3,-1,-4a_4,-1,\cdots,-4a_n), & \text{if } \epsilon_2=-1
\end{cases}\\
\cong\ & \begin{cases}
C(2,-2,\cdots,2,-2+4a_3,0,\underbrace{-2,2,\cdots,-2}_{|4a_4|-1},0,\cdots,-4a_n),\\
\text{if } (\epsilon_2,\epsilon_4)=(1,1);\\[1ex]
C(2,-2,\cdots,2,-2+4a_3,-2,||-4a_4||,-2,\cdots,-4a_n),\\
\text{if } (\epsilon_2,\epsilon_4)=(1,-1);\\[1ex]
C(-2,||-4a_2||,-2,4a_3,0,\underbrace{-2,2,\cdots,-2}_{|4a_4|-1},0,\cdots,4a_n),\\
\text{if } (\epsilon_2,\epsilon_4)=(-1,1);\\[1ex]
C(-2,||-4a_2||,-2,4a_3,-2,||-4a_4||,-2,\cdots,4a_n),\\
\text{if } (\epsilon_2,\epsilon_4)=(-1,-1),
\end{cases}
\end{aligned}
$$

where $\epsilon_i=-a_i/|a_i|$ and $||a||=(\underbrace{-2\epsilon,2\epsilon,\cdots,(-1)^{|a|-1}2\epsilon}_{|a|-1})$. So C is not a torus knot.

Case 2. Let $K=J(\alpha,\beta,\gamma)$, where $\beta\neq 0$ or $\gamma\neq 0$.

From Figure 1b, we see that $b(K)\leq 3$. Next we show that $b(K)\geq 3$. In the same way as in Case 1, we consider the two-fold branched covering space $\Sigma_2(K)$

of S^3 branched over K which is obtained by Dehn surgery of type $p/2$ on the 2-bridge knot $C = C(1, -4, -1, -4\beta, -1, -4\gamma)$, where p is an odd integer. Using Proposition 3, we deform C as follows:

$$\begin{array}{rl} & C(1, -4, -1, -4\beta, -1, -4\gamma) \\ \cong & C(0, 2, -2, 2, -2, -4\beta, -1 + \epsilon_\gamma, || - 4\gamma||) \\ \cong & C(-2, 2, -2, -4\beta, -1 + \epsilon_\gamma, || - 4\gamma||) \\ \ncong & \pm C(2, -2, \cdots, 2, -2), \end{array} \tag{3}$$

where $\epsilon_\gamma = -\gamma/|\gamma|$ and $|| - 4\gamma|| = \underbrace{(-2\epsilon_\gamma, 2\epsilon_\gamma, \cdots, (-1)^{4|\gamma|-1}\epsilon_\gamma)}_{4|\gamma|-1}$.

From this, C is not a torus knot. Hence $\pi_1(\Sigma_2(K))$ is non-cyclic, and thus $b(K) \geq 3$. □

3. Alexander polynomial

Now we calculate the Alexander polynomials of the above two classes of knots.

Case 1. $K(\alpha_1, \cdots, \alpha_n)$. We deform $K(\alpha_1, \cdots, \alpha_n)$ as illustrated in Figures 6a–6d. Thus $(S^3, K(\alpha_1, \cdots, \alpha_n))$ is homeomorphic to $(T(1), K)$ in Figures 6e, where $T(1)$ is obtained by performing the Dehn surgery of type +1 in T (cf. [17]).

To calculate $\Delta_K(t)$, the Alexander polynomial of K, we use the infinite cyclic covering space $\widetilde{E}(K)$ of the exterior of K, $E(K)$; see Chapter 7 in [16]. The Alexander module of K, $H_1(\widetilde{E}(K); Z)$, is a $Z[t, t^{-1}]$-module, where t acts in $\widetilde{E}(K)$ as a covering translation. Let $\{\widetilde{T}_i\}_{i \in Z}$ be the system of the lifts of T in $\widetilde{E}(K)$, and c_0 the surgery coefficient of $\widetilde{T}_i$. Then $\widetilde{T}_j = t^j \widetilde{T}_0$. We put $c_j = lk(\widetilde{T_0}, \widetilde{T_j})$ $(j \neq 0)$. Then $c_j = c_{-j}$ for $j \geq 1$. Now $\widetilde{E}(K)$ is constructed from $\overset{\circ}{D}{}^2 \times R^1$ by removing the regular neighborhoods of $\{\widetilde{T}_i\}_{i \in Z}$ and replacing them with solid tori, where the surgery coefficients are c_0. Then we have

$$\begin{array}{rcl} \Delta_K(t) & = & 1 - 2\alpha_1 + \alpha_1(t + t^{-1}) + \alpha_2(t^2 - 2t + 2 - 2t^{-1} + t^{-2}) \\ & & + \displaystyle\sum_{i=3}^{n} \alpha_i (t^i - 2t^{i-1} + t^{i-2} + t^{-i+2} - 2t^{-i+1} + t^{-i}). \end{array} \tag{4}$$

Let $(\alpha_1, \cdots, \alpha_n)$ be the solution of $\begin{cases} c_n = \alpha_n, \\ c_{n-1} = \alpha_{n-1} - 2\alpha_n, \\ c_i = \alpha_i - 2\alpha_{i+1} + \alpha_{i+2} \ (1 \leq i \leq n-2), \\ c_0 = 1 - 2\sum_{j=1}^{n} c_j. \end{cases}$

Then, for any polynomial $A(t) = \displaystyle\sum_{j=-n}^{n} c_j t^j$ with $A(1) = 1$ and $c_j = c_{-j} (j \geq 1)$, the Alexander polynomial of $K(\alpha_1, \cdots, \alpha_n)$ is equal to $A(t)$.

As an example, we show it for the knot $K(-1, 1, -2)$ given in Figure 7. The infinite cyclic covering space is given in Figure 8.

Case 2. $J(\alpha, \beta, \gamma)$. We calculate the Alexander polynomial of $J(\alpha, \beta, \gamma)$ in the parallel way as in Case 1. $(S^3, J(\alpha, \beta, \gamma))$ is homeomorphic to $(T(1), K)$ in Figure 9c, where $T(1)$ is obtained by perfoming the Dehn surgery of type +1 in T. The infinite cyclic covering space of $\widetilde{E}(J(\alpha, \beta, \gamma))$ is shown in Figure 10, where $\{T_i\}_{i \in Z}$

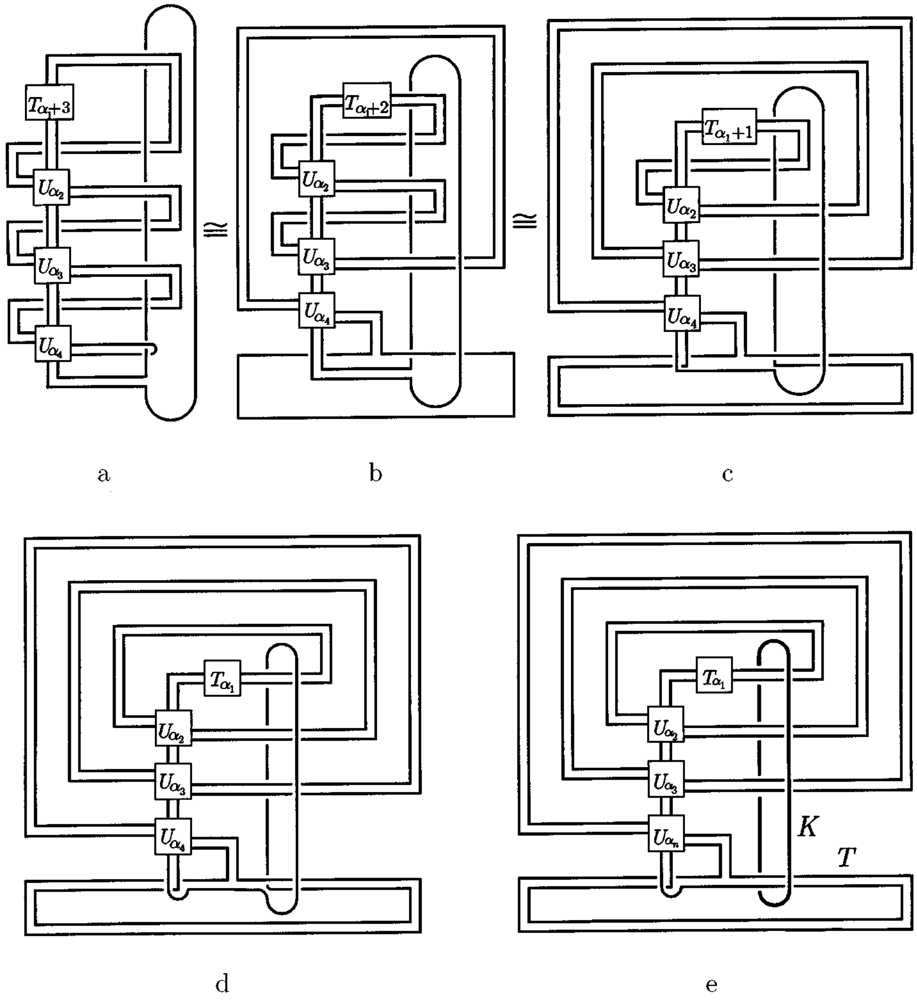


FIGURE 6

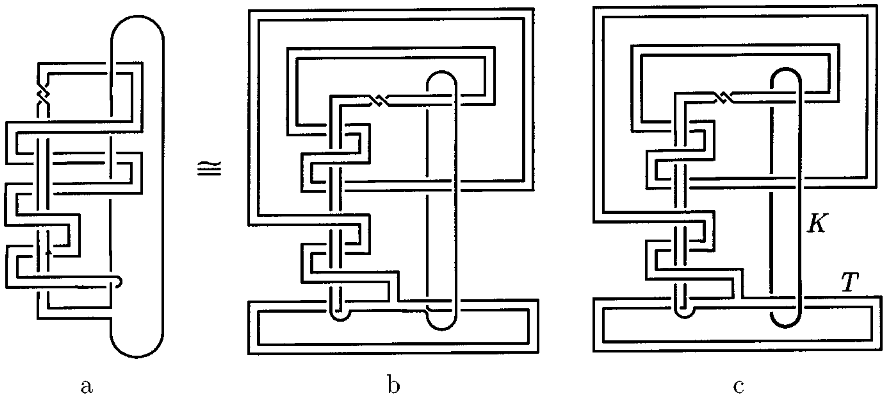


FIGURE 7

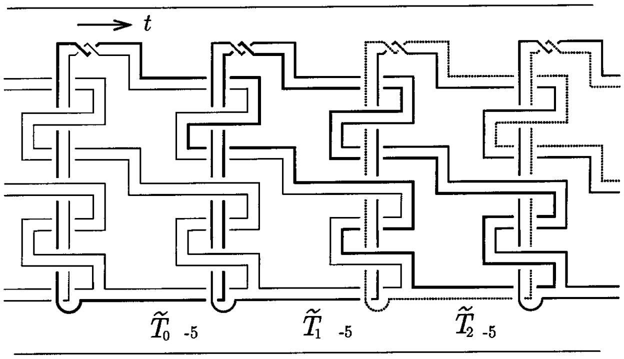
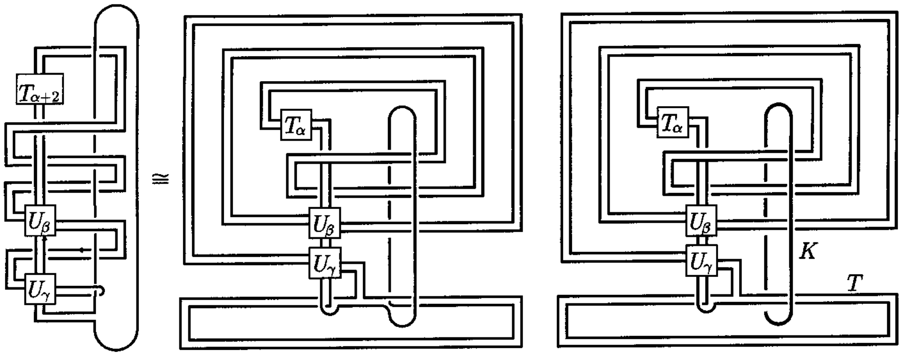


FIGURE 8



a b c

FIGURE 9

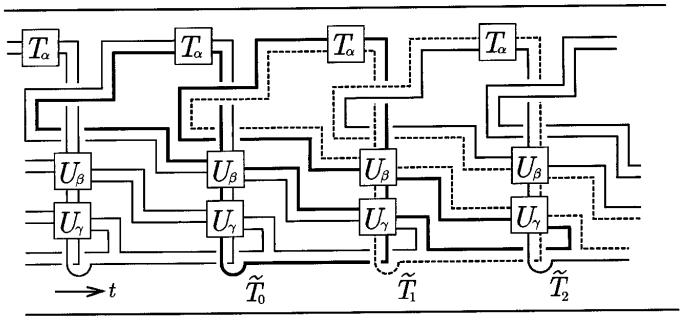


FIGURE 10

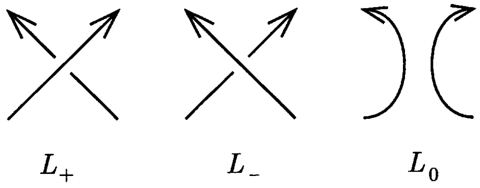


FIGURE 11

is the system of the lifts of T. We see that

(i) $lk(\widetilde{T_0}, \widetilde{T_1}) = \alpha + 2\beta - 2\gamma - 1$,
(ii) $lk(\widetilde{T_0}, \widetilde{T_2}) = 1 + \gamma$, and
(iii) $lk(\widetilde{T_0}, \widetilde{T_i}) = 0$ for $i \geq 3$.

Therefore, we have

$$\Delta_{J(\alpha,\beta,\gamma)}(t) \doteq 2\gamma - 2\alpha - 4\beta + 1 + (\alpha + 2\beta - 2\gamma - 1)(t + t^{-1}) + (\gamma + 1)(t^2 + t^{-2}).$$

If $A(t) = \sum_{i=-2}^{2} c_i t^i$, where $c_i = c_{-i}$ and $A(1) = 1$, then the Alexander polynomial of $J(\alpha, \beta, \gamma)$ is equal to $A(t)$, where $(\alpha, \beta, \gamma) = (c_1 + 2c_2 + 1, -1, c_2 - 1)$. In particular, $\Delta(J(-2\beta - 1, \beta, -1)) = 1$. We denote $J(-2\beta - 1, \beta, -1)$ by K_β.

To prove Lemma 4 below, we introduce a Laurent polynomial invariant ([7], Theorem 1.1), $c_0(K; x)$, determined by the following:

(i) $c_0(O; x) = 1, xc_0(L_+; x) - c_0(L_-; x) = c_0(L_0; x)$, where O is a trivial knot, L_+, L_- are knots and L_0 is a 2-component link, which are identical except near one point where they are as in Figure 11.

(ii) If $L = L_1 \cup L_2$ is a 2-component link with linking number λ, then

$$c_0(L; x) = (x - 1)x^{-\lambda} c_0(L_1; x) c_0(L_2; x).$$

This polynomial is a version of the first term of the skein polynomial [4], [11], [15].

Lemma 4. *Each K_β ($\beta \in Z$) is non-trivial and $K_i \not\cong K_j$ if $i \neq j$.*

Proof. From Figure 12, we calculate the Laurent polynomial invariant of K_β, $c_0(K_\beta; x)$. From (i), we have

$$xc_0(K_\beta; x) - 1 = (x - 1)x^0 c_0(O; x) c_0(C(2, -2, -2, -2\beta, -2, 2); x),$$

and so, we have

$$\begin{aligned} c_0(K_\beta; x) &= x^{-1} + (x - 1)x^{-1} c_0(C(2, -2, -2, -2\beta, -2, 2); x) \\ &= x^{-3} - 3x^{-2} + 3x^{-1} + x^{\beta-2} - 3x^{\beta-1} + 3x^{\beta} - x^{\beta+1}, \end{aligned} \tag{5}$$

from which we obtain the result. □

Concluding remark. From Proposition 1.3 of [14], our two classes of knots have $(1, 1)$-decompositions. If a knot has a $(1, 1)$-decomposition, then its tunnel number is less than or equal to 1. The arc τ^* in Figure 3 is also an unknotting tunnel.

From the results of [9], [19], [20], an unknotting number 1 knot K with $b(K) \leq 3$ is a hyperbolic knot or a trefoil knot. So our knots are hyperbolic knots.

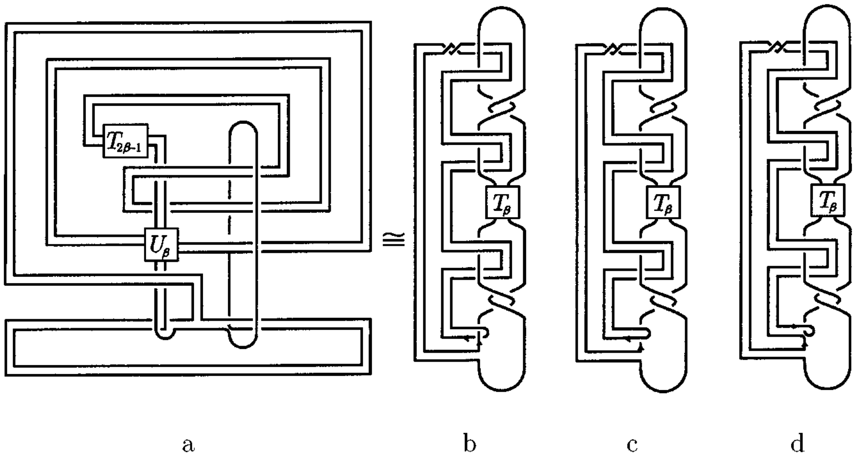


FIGURE 12

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