

ON FULLY INVARIANT IDEALS OF THE FREE GROUP ALGEBRA

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ABSTRACT. Let RF be the group algebra of a free noncyclic group F over an integral domain R . It is proved that if R is not a field, then there exists a fully invariant ideal I of RF such RF/I is torsion-free but not projective as an R -module. In other words, there exists a pure nonprojective variety of group representations over R .

1. INTRODUCTION

Throughout this paper R is a commutative integral domain, not a field. Let F be a free group of countable rank with free generators x_1, x_2, \dots , and let RF be the group algebra of F over R . An ideal I of RF is called *fully invariant*, or *verbal*, if it is invariant under all endomorphisms of F . We will be concerned with the following problem: *does there exist a fully invariant ideal I of RF such that the R -module RF/I is torsion-free but not projective?*

This problem was first posed by Plotkin [P, Problem 15] for $R = \mathbb{Z}$, the ring of integers, and later was extended to arbitrary integral domains [PV, Problem 18.2.8]. It originated in the theory of varieties of group representations. Recall that there is a one-to-one correspondence between the varieties of group representations over R and the fully invariant ideals of RF (for details see [PV] or [V1]). The fully invariant ideal corresponding to a variety \mathcal{X} consists of all *identities* of \mathcal{X} and is denoted by $\text{Id } \mathcal{X}$. A variety \mathcal{X} is called *pure* if $RF/\text{Id } \mathcal{X}$ is a torsion-free R -module, and is called *projective* if $RF/\text{Id } \mathcal{X}$ is a projective R -module. Thus the above problem can be reformulated as follows: *does there exist a pure nonprojective variety of group representations over R ?*

The first example of a pure nonprojective variety was found by Vovsi [V2] over the ring $R = k[t^2, t^3]$ where k is a field of characteristic $\neq 2, 3$. Recently Krasil'nikov [K] showed that a pure nonprojective variety exists over the ring of integers as well. Completing this work, in the present note we solve the problem over an *arbitrary* integral domain.

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Theorem. *Let R be an arbitrary integral domain, not a field, and let F be a free group of rank ≥ 2 . Then there exists a fully invariant ideal I of RF such that the R -module RF/I is torsion-free but not projective.*

The method of the proof is close to that in [K] and is essentially based on the following surprising result of Storozhev [S]: *There exist a 2-generated relatively free group G and a verbal subgroup V of G such that:*

- (i) V is a free abelian group of countable rank;
- (ii) V is contained in the center of G ;
- (iii) for every endomorphism ϕ of G there exists a natural number $n = n(\phi)$ such that $v^\phi = v^n$ for all $v \in V$.

Note that the property (iii) is not explicitly stated in [S] but follows from the constructions of that paper. Using Storozhev's theorem, we first prove the following statement which is of independent interest.

Proposition 1. *Let R be an integral domain, F_2 a free group of rank 2, and M an arbitrary countably generated R -module. Then there exists a fully invariant ideal J of RF_2 such that*

$$RF_2/J \simeq M \oplus \Phi$$

where Φ is the free R -module of countable rank.

Our theorem is then easily deduced from this statement and one simple fact of commutative algebra. On the other hand, in the end of the paper we give an example of a condition sufficient for a pure variety to be projective. Namely, we prove that if R is a PID, then every pure variety of *ring type* over R must be projective.

We would like to express our appreciation to the referee whose excellent report has led to a number of improvements in this paper.

2. PROOF OF THE THEOREM

For any group G , the augmentation ideal of the group algebra RG is denoted by Δ_G . The following fact is, probably, well known; a simple proof can be found, for example, in [K].

Lemma 1. *If H is a subgroup of a group G , then*

$$\Delta_H \cap \Delta_G \Delta_H = \Delta_H^2. \quad \square$$

Now we fix a group G and a verbal subgroup V of G satisfying the conditions of Storozhev's theorem. Then $\Delta_G \Delta_V = \Delta_G(V - 1)$ is a fully invariant ideal of the group algebra RG .

Lemma 2. (a) *There is an isomorphism of R -modules*

$$RG \cdot \Delta_V / \Delta_G \Delta_V \simeq \Delta_V / \Delta_V^2.$$

(b) $RG \cdot \Delta_V / \Delta_G \Delta_V$ is a free R -module of countable rank.

(c) *Every endomorphism ϕ of G acts on $RG \cdot \Delta_V / \Delta_G \Delta_V$ as a multiplication by some natural number $n = n(\phi)$.*

Proof. (a) Since $RG \cdot \Delta_V = (R \cdot 1 \oplus \Delta_G)\Delta_V = \Delta_V + \Delta_G\Delta_V$, we have

$$RG \cdot \Delta_V / \Delta_G\Delta_V = (\Delta_V + \Delta_G\Delta_V) / \Delta_G\Delta_V \simeq \Delta_V / (\Delta_V \cap \Delta_G\Delta_V),$$

and it remains to apply Lemma 1.

(b) It is well known that if V is a free abelian group with a basis $\{v_i\}$, then Δ_V / Δ_V^2 is a free R -module with the basis $\{v_i - 1\}$. The rest follows from (a).

(c) An arbitrary element of $RG \cdot \Delta_V / \Delta_G\Delta_V$ is a sum of elements

$$a = u(v - 1) + \Delta_G\Delta_V \quad (u \in RG, v \in V).$$

If $\epsilon : RG \rightarrow R$ is the augmentation homomorphism (sending all elements of G to 1), then

$$\begin{aligned} a &= u(v - 1) + \Delta_G\Delta_V \\ &= [u^\epsilon + (u - u^\epsilon)](v - 1) + \Delta_G\Delta_V \\ &= u^\epsilon(v - 1) + (u - u^\epsilon)(v - 1) + \Delta_G\Delta_V \\ &= u^\epsilon(v - 1) + \Delta_G\Delta_V \end{aligned}$$

(since $u - u^\epsilon \in \Delta_G$). By (iii), for every $\phi \in \text{End } G$ there exists $n = n(\phi)$ such that $v^\phi = v^n$ for all $v \in V$. Therefore

$$a^\phi = u^\epsilon(v^\phi - 1) + \Delta_G\Delta_V = u^\epsilon(v^n - 1) + \Delta_G\Delta_V.$$

Since $v^n - 1 \equiv n(v - 1) \pmod{\Delta_V^2}$, we have

$$a^\phi = u^\epsilon n(v - 1) + \Delta_G\Delta_V = na,$$

as required. □

Lemma 3. *Every R -submodule A of RG such that $RG \cdot \Delta_V \supseteq A \supseteq \Delta_G\Delta_V$ is a fully invariant ideal of RG .*

Proof. By Lemma 2(c), A is invariant under all endomorphisms of G . Let us show that A is an ideal of RG , that is, A is closed under left and right multiplications by elements of G . Take $a \in A$, $g \in G$. Since $a \in RG \cdot \Delta_V$, we have $(g - 1)a \in \Delta_G\Delta_V$ and so $ga = (g - 1)a + a \in A$. As to right multiplications, we have $ag = g(a^\phi)$ where ϕ is the conjugation by g . By Lemma 2(c), $a^\phi = na$ for some natural n , and so $ag = g(na) = n(ga) \in A$. □

Proof of Proposition 1. Let G and V be the same as before. By Lemma 2(b), $RG \cdot \Delta_V / \Delta_G\Delta_V$ is a free R -module of countable rank. Therefore there exists an R -module A such that

$$RG \cdot \Delta_V \supseteq A \supseteq \Delta_G\Delta_V$$

and $RG \cdot \Delta_V / A \simeq M$. By the preceding lemma A is a fully invariant ideal of RG . Furthermore, RG/A is an extension of $RG \cdot \Delta_V / A \simeq M$ by $RG / RG \cdot \Delta_V$. The latter is isomorphic to the group algebra $R[G/V]$ and therefore is isomorphic to Φ as an R -module. Every extension by a free module splits, hence $RG/A \simeq M \oplus \Phi$.

Now we recall that G is a 2-generated relatively free group. Therefore $G = F_2/H$ for some verbal subgroup H of F_2 . Let $\kappa : RF_2 \rightarrow RG = R[F_2/H]$ be the canonical epimorphism. Since A is a fully invariant ideal of RG , it follows that $J = \kappa^{-1}(A)$ is a fully invariant ideal of RF_2 , and $RF_2/J \simeq RG/A \simeq M \oplus \Phi$. □

The following fact is certainly well known, but we could not find an explicit reference.

Lemma 4. *Let R be an integral domain which is not a field, and let a be a nonzero noninvertible element of R . Then $R[a^{-1}]$ is nonprojective as an R -module.*

Proof. Let \mathfrak{m} be a maximal ideal of R containing a . Then the localization $R_{\mathfrak{m}}$ is a local ring, and so all projective modules over $R_{\mathfrak{m}}$ are free. If $R[a^{-1}]$ were a projective R -module, then by localizing at \mathfrak{m} it would follow that $R_{\mathfrak{m}}[a^{-1}]$ was a projective $R_{\mathfrak{m}}$ -module, hence a free $R_{\mathfrak{m}}$ -module. But since a is not invertible in $R_{\mathfrak{m}}$, every nonzero free $R_{\mathfrak{m}}$ -module has elements not divisible by a , which is not true of $R_{\mathfrak{m}}[a^{-1}]$. \square

Proof of Theorem. Let F_2 be the free group of rank 2. Since R is not a field, Lemma 4 guarantees that there exists a countably generated torsion-free R -module M which is not projective. By Proposition 1, there is a fully invariant ideal J of RF_2 such that $RF_2/J \simeq M \oplus \Phi$. Since M is not projective as an R -module, neither is RF_2/J .

Consider RF_2/J as an RF_2 -module, and let $\rho = (RF_2/J, F_2)$ be the corresponding representation of the group F_2 . Denote by \mathcal{X} the variety of group representations generated by ρ . Since RF_2/J is torsion-free over R , \mathcal{X} is a pure variety, that is, the modules of all free representations of \mathcal{X} are R -torsion-free.

Now let F be a free group of rank > 2 . Denote by I the verbal ideal of RF corresponding to the variety \mathcal{X} . Then $(RF/I, F)$ is a free representation of \mathcal{X} and hence RF/I is R -torsion-free. A standard argument in variety theory shows that $I \cap RF_2 = J$ (cf. for instance [N, 12.63]). It follows that the map α such that $x_1^\alpha = x_1$, $x_2^\alpha = x_2$, $x_i^\alpha = 1$ for $i > 2$ induces an epimorphism

$$\alpha : RF/I \rightarrow RF_2/J$$

of R -modules (and even of R -algebras). This epimorphism is obviously a retraction because the natural embedding $\beta : RF_2/J \rightarrow RF/I$ satisfies the condition $\beta\alpha = 1$. Therefore RF_2/J is a direct summand of RF/I . Since the former is not projective, neither is the latter. \square

3. WHEN IS A PURE VARIETY PROJECTIVE?

We have shown that a pure variety of group representations, in general, need not be projective. Are there any special classes of varieties for which the implication “pure \implies projective” is valid? One such class is considered in this section.

First we recall a natural connection between varieties of associative algebras over R and varieties of group representations over the same R (for details see [V1, Section 1.3]). Let $R\langle\langle Z \rangle\rangle$ be the algebra (with 1) of formal power series over R in a countable set of noncommuting variables $Z = \{z_1, z_2, \dots\}$. The elements $1 + z_i$ are invertible in $R\langle\langle Z \rangle\rangle$ and, as discovered by Magnus [M], the map $x_i \mapsto 1 + z_i$ can be uniquely extended to a monomorphism of R -algebras $RF \rightarrow R\langle\langle Z \rangle\rangle$. In the sequel we will identify RF with its image in $R\langle\langle Z \rangle\rangle$, so that every $u \in RF$ can be uniquely written as a formal power series

$$u = u_{(0)} + u_{(1)} + \dots + u_{(n)} + \dots$$

where $u_{(n)} = \sum \lambda_{i_1 \dots i_n} z_{i_1} \dots z_{i_n}$, the homogeneous component of degree n of u , is a finite R -linear combination of monomials of degree n in the z_i . Furthermore, the image of RF in $R\langle\langle Z \rangle\rangle$ naturally contains the free associative algebra $R\langle Z \rangle$

(without 1!), so we have an embedding of R -algebras ¹

$$R\langle Z \rangle \subset RF \subset R\langle\langle Z \rangle\rangle.$$

Now let \mathcal{M} be a variety of (associative) algebras over R , and let $T = T(\mathcal{M})$ be the corresponding fully invariant ideal (or T-ideal) of $R\langle Z \rangle$. Set

$$(1) \quad \alpha T = \{ u \in RF \mid \forall n : u_{(n)} \in T \}.$$

Note that T and αT have the same homogeneous elements. One can prove that αT is a verbal ideal of KF , so that it determines a variety of group representations which we denote by $\alpha\mathcal{M}$. The map $\mathcal{M} \mapsto \alpha\mathcal{M}$ yields the desired connection between the varieties of algebras and varieties of group representations over the ring R . We will say that the varieties of the form $\alpha\mathcal{M}$ (and the corresponding verbal ideals αT) are of *ring type*.

Proposition 2. *Let R be a principal ideal domain. Then every pure variety of ring type over R is projective.*

Proof. We assume that R is infinite, for otherwise R is a field and there is nothing to prove. Let T be a T-ideal of $R\langle Z \rangle$ and let $I = \alpha T$ be the corresponding verbal ideal of RF . Assume that RF/I is torsion-free over R . We have to show that RF/I is a projective (actually free) R -module.

It follows from (1) that I has the following property: for any $u \in RF$

$$(2) \quad u \in I \iff \forall n : u_{(n)} \in I.$$

Moreover, we will show that I is a *multihomogeneous* ideal, that is, for every $u \in I$ all of its multihomogeneous components (i.e. homogeneous in each variable z_i) are also elements of I . In view of (2), it suffices to consider the case when u is homogeneous. Then $u \in T$. Let v be a multihomogeneous component of u . Since T is a T-ideal of $R\langle Z \rangle$ and R is infinite, a standard Vandermonde determinant argument implies that $rv \in T$ for some nonzero $r \in R$ (see for example [ZhS, Section 1.3]). Being a homogeneous element, rv must belong to I . Since RF/I is torsion-free, $v \in I$, as required.

Denote the power series ring $R\langle\langle Z \rangle\rangle$ by S , and for every family of indices $\lambda = \{i_1, \dots, i_n\}$, perhaps with repetitions, denote by S_λ the corresponding multihomogeneous component of S . Each S_λ is a free R -module of finite rank (for example, if $\lambda = \{1, 1, 2\}$, then S_λ is a free R -module with basis $z_1^2 z_2, z_1 z_2 z_1, z_2 z_1^2$). Let $S_{(n)}$ be the homogeneous component of degree n of S , that is, the R -span of all monomials of degree n . Then S is the direct *product* of all $S_{(n)}$ and each $S_{(n)}$ is the direct *sum* of all S_λ with $|\lambda| = n$. Hence

$$RF \subset S = \prod_n \left(\bigoplus_{|\lambda|=n} S_\lambda \right).$$

Let \hat{T} be the submodule of S consisting of the elements all of whose multihomogeneous components belong to T . It follows from (2) and multihomogeneity of I that $\hat{T} \cap RF = I$, and that

$$\hat{T} = \prod_n \left(\bigoplus_{|\lambda|=n} I_\lambda \right)$$

¹We emphasize that under this embedding the free generators z_i of $R\langle Z \rangle$ are identified with the elements $x_i - 1$, where x_i are the free generators of F .

where $I_\lambda = I \cap S_\lambda$ (note that $I \cap S_\lambda = T \cap S_\lambda$ for every S_λ). Therefore

$$RF/I = RF/(\hat{T} \cap RF) = (RF + \hat{T})/\hat{T} \subset S/\hat{T} = \prod_n \left(\bigoplus_{|\lambda|=n} (S_\lambda/I_\lambda) \right).$$

Since $S_\lambda \subset RF$ and RF/I is torsion-free, $S_\lambda/I_\lambda \simeq (S_\lambda + I)/I$ is torsion-free as well. Being a finitely generated torsion-free module over a PID, S_λ/I_λ is a free module. Thus RF/I is a countably generated R -submodule of a direct product of free R -modules. It is well known that if R is a PID, then such a submodule must be free, as required. \square

As noted by the referee, this proof can be generalized to the case when R is a Dedekind domain.

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