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THE DISTANCE FROM THE APOSTOL SPECTRUM

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Abstract. If T is an s-regular operator in a Banach space (i.e. T has closed range and $N(T) \subset R^\infty(T)$) and $\gamma(T)$ is the Kato reduced minimum modulus, then

$$\lim_{n\to\infty} \gamma(T^n)^{1/n} = \sup\{r : T-\lambda \text{ is s-regular for } |\lambda| < r\}.$$

Let x be an element of a Banach algebra A. The spectral radius of x is given by the well-known spectral radius formula: $r(x) = \lim_{n\to\infty} \|x^n\|^{1/n}$.

There are a number of generalizations of this formula. If we set $d(x) = \inf\{\|xy\| : y \in A, \|y\| = 1\}$ and denote by $\tau_l(x) = \{\lambda \in \mathbf{C} : d(x-\lambda) = 0\}$ the left approximate point spectrum of x, then $\operatorname{dist}\{0, \tau_l(x)\} = \lim_{n\to\infty} d(x^n)^{1/n}$; see [13], [9]. In particular, in the algebra $B(X)$ of all bounded linear operators in a Banach space X this gives formulas for radii of boundedness below or surjectivity:

$$\sup\{r : T-\lambda \text{ is bounded below for } |\lambda| < r\} = \lim_{n\to\infty} j(T^n)^{1/n}$$

and

$$\sup\{r : T-\lambda \text{ is onto for } |\lambda| < r\} = \lim_{n\to\infty} k(T^n)^{1/n},$$

where $j(T)$ and $k(T)$ are the moduli of injectivity and surjectivity of T:

$$j(T) = \inf\{\|Tx\| : x \in X, \|x\| = 1\} \quad \text{and} \quad k(T) = \sup\{r : TU_X \supset rU_X\},$$

where U_X is the closed unit ball in X.

For a bounded linear operator T in a Banach space X denote by $N(T)$ and $R(T)$ its kernel and range, respectively. Denote further $R^\infty(T) = \bigcap_{n=1}^\infty R(T^n)$ and $N^\infty(T) = \bigcup_{n=1}^\infty N(T^n)$.

The injectivity and surjectivity moduli of an operator which is bounded below (onto) are special cases of the Kato reduced minimum modulus [7]

$$\gamma(T) = \inf\Big\{\frac{\|Tx\|}{\operatorname{dist}\{x, N(T)\}} : x \in X \backslash N(T)\Big\}$$

(for $T = 0$ we define formally $\gamma(T) = \infty$).

The existence and the meaning of the limit $\lim_{n\to\infty} \gamma(T^n)^{1/n}$ in a more general setting were studied by Apostol [1] and Mbekhta [10].

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Definition. Let $T \in B(X)$. We say that T is s-regular (= semi-regular) if $R(T)$ is closed and $N(T) \subset R^\infty(T)$.

The s-regular operators and closely related classes of operators were studied (under various names) by many authors; see [3, 4, 5, 6, 8, 16]. We list some of the most important equivalent conditions for s-regular operators; see [11, 12].

Theorem. *Let $T \in B(X)$ be an operator with a close range. The following conditions are equivalent:*

(1) *T is s-regular,*
(2) *the function $\lambda \mapsto R(T-\lambda)$ is continuous at 0 in the gap topology,*
(3) *the function $\lambda \mapsto N(T-\lambda)$ is continuous at 0 in the gap topology,*
(4) *the function $\lambda \mapsto \gamma(T-\lambda)$ is continuous at 0,*
(5) $\liminf_{\lambda\to 0} \gamma(T-\lambda) > 0$,
(6) $N^\infty(T) \subset R(T)$,
(7) $N^\infty(T) \subset R^\infty(T)$.

Denote further $\sigma_\gamma(T) = \{\lambda \in \mathbf{C} : T-\lambda \text{ is not s-regular}\}$. The set $\sigma_\gamma(T)$ was studied by Apostol [1], Rakočevič [15], Mbekhta and Ouahab [11, 12] and Mbekhta [10]. The terminology is not unified; we suggest to call $\sigma_\gamma(T)$ the Apostol spectrum of T.

The Apostol spectrum $\sigma_\gamma(T)$ is always a non-empty compact subset of the complex plane, $\partial\sigma(T) \subset \sigma_\gamma(T) \subset \sigma(T)$ and $\sigma_\gamma f(T) = f\sigma_\gamma(T)$ for any function f analytic in a neighbourhood of $\sigma(T)$.

If T is an s-regular operator in a Hilbert space, then the limit $\lim_{n\to\infty} \gamma(T^n)^{1/n}$ exists and

$$\lim_{n\to\infty} \gamma(T^n)^{1/n} = \operatorname{dist}\{0, \sigma_\gamma(T)\} = \sup\{r : T-\lambda \text{ is s-regular for } |\lambda| < r\}; \tag{1}$$

see [1, Theorem 3.2] or [10, Theorem 3.1].

The aim of this paper is to prove equality (1) for operators in Banach spaces. This gives a positive answer to the conjecture of Rakočevič [15] and Mbekhta and generalizes the above-mentioned results for radii of injectivity and surjectivity.

Further, we study the essential version of this result.

If T is a semi-Fredholm operator, then the limit $\lim_{n\to\infty} \gamma(T^n)^{1/n}$ exists by [2] and it is equal to the semi-Fredholm radius of T:

$$\lim_{n\to\infty} \gamma(T^n)^{1/n} = \sup\{r : T-\lambda \text{ is semi-Fredholm for } |\lambda| < r\};$$

see [17] and [2].

We prove a similar formula for essentially s-regular operators which generalizes the semi-Fredholm case.

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Lemma 1. *$T \in B(X)$ is s-regular if and only if there exists a closed subspace $M \subset X$ such that $TM = M$ and the operator $\tilde{T} : X/M \to X/M$ induced by T is bounded below.*

Proof. If T is s-regular, then set $M = R^\infty(T)$. It is well known that M is closed and (see e.g. [4, Theorem 3.4]) that $TM = M$ and $\tilde{T} : X/M \to X/M$ is bounded from below.

Conversely, let M be the subspace of X with the required properties. Then $TM = M$ implies $M \subset R^\infty(T)$. If $Tx = 0$, then $\tilde{T}(x+M) = 0$ and the injectivity of $\tilde{T}$ implies $x \in M$. Thus $N(T) \subset M \subset R^\infty(T)$.

It remains to prove that T has closed range. Let $\pi : X \to X/M$ be the canonical projection. We show $R(T) = \pi^{-1}R(\tilde{T})$. If $y \in R(T), y = Tx$ for some $x \in X$, then $\pi y = Tx + M = \tilde{T}(x+M) \in R(\tilde{T})$ so that $R(T) \subset \pi^{-1}R(\tilde{T})$. If $y \in X$ and $\pi y \in R(\tilde{T})$, i.e. $y + M = Tx + M$ for some $x \in X$, then $y \in R(T)$ since $M \subset R(T)$. Thus $R(T) = \pi^{-1}R(\tilde{T})$, which is closed since $R(\tilde{T})$ is closed and π continuous.

Lemma 2. *Let $T \in B(X)$, and let M be a closed subspace of X such that $TM = M$ and the operator $\tilde{T} : X/M \to X/M$ induced by T is bounded below. Denote by $T_1 : M \to M$ the restriction of T to M. Then*

$$\lim_{n\to\infty} \gamma(T^n)^{1/n} = \min\bigl\{\lim_{n\to\infty} \gamma(T_1^n)^{1/n}, \lim_{n\to\infty} \gamma(\tilde{T}^n)^{1/n}\bigr\}.$$

Proof. The limits on the right-hand side exist by [17]. If $T^n x = 0$, then $\tilde{T}^n(x+M) = 0$, i.e. $x \in M$. Thus $N(T^n) \subset M$ and $N(T_1^n) = N(T^n)$. We have

$$\begin{aligned}\gamma(T_1^n) &= \inf\Bigl\{\frac{\|T_1^n x\|}{\operatorname{dist}\{x, N(T_1^n)\}} : x \in M\setminus N(T_1^n)\Bigr\}\\ &= \inf\Bigl\{\frac{\|T^n x\|}{\operatorname{dist}\{x, N(T^n)\}} : x \in M\setminus N(T^n)\Bigr\} \ge \gamma(T^n).\end{aligned}$$

Further, since $TM = M$,

$$\begin{aligned}\gamma(\tilde{T}^n) &= \inf\Bigl\{\frac{\|\tilde{T}^n(x+M)\|}{\|x+M\|} : x \notin M\Bigr\} = \inf\Bigl\{\frac{\|T^n x + M\|}{\operatorname{dist}\{x, M\}} : x \notin M\Bigr\}\\ &\ge \inf\Bigl\{\frac{\|T^n x\|}{\operatorname{dist}\{x, M\}} : x \notin M\Bigr\} \ge \inf\Bigl\{\frac{\|T^n x\|}{\operatorname{dist}\{x, N(T^n)\}} : x \notin M\Bigr\} \ge \gamma(T^n).\end{aligned}$$

Thus $\gamma(T^n) \le \min\{\gamma(T_1^n), \gamma(\tilde{T}^n)\}$ and

$$\limsup_{n\to\infty} \gamma(T^n)^{1/n} \le \min\bigl\{\lim_{n\to\infty} \gamma(T_1^n)^{1/n}, \lim_{n\to\infty} \gamma(\tilde{T}^n)^{1/n}\bigr\}.$$

Denote

$$s = \min\bigl\{\lim_{n\to\infty} \gamma(T_1^n)^{1/n}, \lim_{n\to\infty} \gamma(\tilde{T}^n)^{1/n}\bigr\}.$$

We prove $\liminf_{n\to\infty} \gamma(T^n)^{1/n} \ge s$.

Let $n \ge 1$, $x = x_0 \in R(T^n)$, $\|x\| = 1$, and let $s > \varepsilon > 0$. Then $x + M \in R(\tilde{T}^n)$ and

$$\|\tilde{T}^{-i}(x+M)\| \le \gamma(\tilde{T}^i)^{-1}\|x+M\| \le \gamma(\tilde{T}^i)^{-1} \qquad (i = 1, \dots, n).$$

Thus there exist vectors $x_i \in \tilde{T}^{-i}(x+M)$ such that

$$\|x_i\| \le \gamma(\tilde{T}^i)^{-1}(1+\varepsilon) \qquad (i = 1, \dots, n).$$

Denote $m_i = Tx_{i+1} - x_i$ $(i = 0, \dots, n-1)$. Then

$$\|m_i\| \le \|T\|\|x_{i+1}\| + \|x_i\| \le (1+\varepsilon)\bigl[\|T\|\gamma(\tilde{T}^{i+1})^{-1} + \gamma(\tilde{T}^i)^{-1}\bigr] \quad (i = 0, \dots, n-1).$$

Further, $\tilde{T}^i(m_i + M) = T^{i+1}x_{i+1} - T^i x_i + M = M$ so that $m_i \in M$ for each i. We have

$$\sum_{i=0}^{n-1} T^i m_i = (T^n x_n - T^{n-1}x_{n-1}) + (T^{n-1}x_{n-1} - T^{n-2}x_{n-2}) + \cdots + (Tx_1 - x)$$
$$= T^n x_n - x.$$

Since $T_1 M \to M$ is onto, there exist vectors $m_i' \in M$ such that $T^{n-i}m_i' = m_i$ and $\|m_i'\| \le (1+\varepsilon)\gamma(T_1^{n-i})^{-1}\|m_i\|$. Thus

$$T^n\left(x_n - \sum_{i=0}^{n-1} m_i'\right) = T^n x_n - \sum_{i=0}^{n-1} T^i m_i = x$$

and

$$\left\|x_n - \sum_{i=0}^{n-1} m_i'\right\| \le (1+\varepsilon)\gamma(\tilde{T}^n)^{-1} + \sum_{i=0}^{n-1}(1+\varepsilon)^2\gamma(T_1^{n-i})^{-1}\Big[\|T\|\gamma(\tilde{T}^{i+1})^{-1} + \gamma(\tilde{T}^i)^{-1}\Big].$$

Thus

$$\gamma(T^n)^{-1} \le (1+\varepsilon)\gamma(\tilde{T}^n)^{-1} + \sum_{i=0}^{n-1}(1+\varepsilon)^2\gamma(T_1^{n-i})^{-1}\Big[\|T\|\gamma(\tilde{T}^{i+1})^{-1} + \gamma(\tilde{T}^i)^{-1}\Big].$$

Find n_0 such that

$$\gamma(T_1^i) \ge (s-\varepsilon)^i, \quad \gamma(\tilde{T}^i) \ge (s-\varepsilon)^i \qquad (i \ge n_0).$$

Denote

$$K = \max_{1 \le i \le n_0+1} \max\big\{\gamma(T_1^i)^{-1}, \gamma(\tilde{T}^i)^{-1}, (s-\varepsilon)^{-i}\big\}.$$

For n large enough we have

$$\begin{aligned}
\gamma(T^n)^{-1} &\le (1+\varepsilon)^2\Big[(s-\varepsilon)^{-n} + \sum_{i=n_0}^{n-n_0-1}(s-\varepsilon)^{i-n}\big(\|T\|(s-\varepsilon)^{-i-1} + (s-\varepsilon)^{-i}\big) \\
&+ \sum_{i=0}^{n_0-1}(s-\varepsilon)^{i-n}\big(\|T\|\cdot K + K\big) + \sum_{i=n-n_0}^{n-1} K\big(\|T\|(s-\varepsilon)^{-i-1} + (s-\varepsilon)^{-i}\big)\Big] \\
&\le (1+\varepsilon)^2(s-\varepsilon)^{n_0-n}\Big[K + (n-2n_0)(K\cdot\|T\| + K) + 2n_0K(\|T\|\cdot K + K)\Big] \\
&\le (1+\varepsilon)^2(s-\varepsilon)^{n_0-n} n \cdot K',
\end{aligned}$$

where K' is a constant independent of n. Hence

$$\liminf_{n\to\infty} \gamma(T^n)^{1/n} \ge \liminf_{n\to\infty}(s-\varepsilon)^{\frac{n-n_0}{n}} = s-\varepsilon.$$

Since $\varepsilon > 0$ was arbitrary, we conclude that $\liminf_{n\to\infty}\gamma(T^n)^{1/n} \ge s$, so that

$$\lim_{n\to\infty}\gamma(T^n)^{1/n} = s.$$

Theorem 3. *Let $T \in B(X)$ be s-regular. Then*

$$\operatorname{dist}\{0, \sigma_\gamma(T)\} = \lim_{n\to\infty} \gamma(T^n)^{1/n}.$$

Proof. Denote $r = \operatorname{dist}\{0, \sigma_\gamma(T)\}$. Let $M = R^\infty(T), T_1 = T|M$, and let $\tilde{T} : X/M \to X/M$ be the operator induced by T. If λ is a complex number satisfying

$$|\lambda| < \lim_{n\to\infty} \gamma(T^n)^{1/n} = \min\{\lim_{n\to\infty} \gamma(T_1^n)^{1/n}, \lim_{n\to\infty} \gamma(\tilde{T}^n)^{1/n}\},$$

then $T_1 - \lambda$ is onto and $\tilde{T} - \lambda$ is bounded below. Thus $T - \lambda$ is s-regular by Lemma 1 and $\lim_{n\to\infty} \gamma(T^n)^{1/n} \le r$.

Conversely, it is well known (see e.g. [15, Theorem 5.2]) that $R^\infty(T - \lambda)$ is constant on the component of $\mathbf{C}\backslash\sigma_\gamma(T)$ containing 0, in particular $R^\infty(T-\lambda) = M$ for $|\lambda| < r$. If $|\lambda| < r$, then $(T - \lambda)M = M$ and $\tilde{T} - \lambda = \widetilde{T - \lambda} : X/M \to X/M$ is bounded below. Thus $\lim_{n\to\infty} \gamma(T_1^n)^{1/n} \ge r$ and $\lim_{n\to\infty} \gamma(\tilde{T}^n)^{1/n} \ge r$. Hence $\lim_{n\to\infty} \gamma(T^n)^{1/n} \ge r$ by Lemma 2.

Remark. It is possible to deduce the inequality $\operatorname{dist}\{0, \sigma_\gamma(T)\} \ge \lim_{n\to\infty} \gamma(T^n)^{1/n}$ from [11, Theorem 2.10]. We have obtained a new direct proof of this result.

Definition. $T \in B(X)$ is called essentially s-regular if $R(T)$ is closed and there exists a finite-dimensional subspace $F \subset X$ such that $N(T) \subset R^\infty(T) + F$.

Define further $\sigma_{e\gamma}(T) = \{\lambda \in \mathbf{C} : T - \lambda \text{ is not essentially s-regular}\}$.

For properties of essentially s-regular operators and the set $\sigma_{e\gamma}(T)$ see [14, 15].

Theorem 4. *Let $T \in B(X)$ be essentially s-regular. Then $\lim_{n\to\infty} \gamma(T^n)^{1/n}$ exists and*

$$\begin{aligned}\lim_{n\to\infty} \gamma(T^n)^{1/n} &= \max\bigl\{r : T - \lambda \textit{ is s-regular for } 0 < |\lambda| < r\bigr\} \\ &= \operatorname{dist}\bigl\{0, \sigma_\gamma(T)\backslash\{0\}\bigr\}.\end{aligned}$$

Proof. By [14, Theorem 3.1] or [15, Theorem 2.1] there exist subspaces $X_1, X_2 \subset X$ such that $X = X_1 \oplus X_2, \dim X_1 < \infty, TX_1 \subset X_1, TX_2 \subset X_2, T_1 = T|X_1$ is nilpotent and $T_2 = T|X_2$ is s-regular (the Kato decomposition). By the previous theorem $\operatorname{dist}\{0, \sigma_\gamma(T_2)\} = \lim_{n\to\infty} \gamma(T_2^n)^{1/n}$. For $n \ge \dim X_1$ we have $T_1^n = 0$ so that $N(T^n) = X_1 \oplus N(T_2^n)$. Let P be the projection with $R(P) = X_2$ and $N(P) = X_1$. Let $x_2 \in X_2$. We have

$$\begin{aligned}\operatorname{dist}\{x_2, N(T_2^n)\} =& \inf\{\|x_2 - y_2\| : y_2 \in X_2, T_2^n y_2 = 0\} \\ \le& \|P\| \inf\{\|y_1 \oplus (x_2 - y_2)\| : y_1 \in X_1, y_2 \in X_2, T_2^n y_2 = 0\} \\ =& \|P\| \operatorname{dist}\{x_2, N(T^n)\} \le \|P\| \operatorname{dist}\{x_2, N(T_2^n)\}.\end{aligned}$$

Then

$$\begin{aligned}\gamma(T_2^n) =& \inf\Bigl\{\frac{\|T_2^n x_2\|}{\operatorname{dist}\{x_2, N(T_2^n)\}} : x_2 \in X_2\backslash N(T_2^n)\Bigr\} \\ \le& \inf\Bigl\{\frac{\|T^n x_2\|}{\operatorname{dist}\{x_2, N(T^n)\}} : x_2 \in X_2\backslash N(T^n)\Bigr\} \\ =& \inf\Bigl\{\frac{\|T^n (x_1 \oplus x_2)\|}{\operatorname{dist}\{x_1 \oplus x_2, N(T^n)\}} : x_1 \oplus x_2 \in X\backslash N(T^n)\Bigr\} = \gamma(T^n)\end{aligned}$$

and

$$\gamma(T^n) \le \inf\Big\{\frac{\|T_2^n x_2\|}{\operatorname{dist}\{x_2, N(T^n)\}} : x_2 \in X_2 \backslash N(T_2^n)\Big\}$$
$$\le \|P\| \inf\Big\{\frac{\|T_2^n x_2\|}{\operatorname{dist}\{x_2, N(T_2^n)\}} : x_2 \in X_2 \backslash N(T_2^n)\Big\} = \|P\|\gamma(T_2^n).$$

Hence $\lim_{n\to\infty} \gamma(T^n)^{1/n} = \lim_{n\to\infty} \gamma(T_2^n)^{1/n}$.

If $\lambda \neq 0$, then $T-\lambda$ is s-regular if and only if $T_2 - \lambda$ is s-regular. Then

$$\max\{r : T-\lambda \text{ is s-regular for } 0<|\lambda|<r\} = \operatorname{dist}\{0, \sigma_\gamma(T_2)\} = \lim_{n\to\infty} \gamma(T^n)^{1/n}.$$

The following lemma is an analog of Lemma 1 for essentially s-regular operators:

Lemma 5. $T \in B(X)$ *is essentially s-regular if and only if there exists a closed subspace* $M \subset X$ *such that* $TM = M$ *and the operator* $\tilde T : X/M \to X/M$ *induced by* T *is upper semi-Fredholm.*

Proof. If T is essentially s-regular, then set $M = R^\infty(T)$. If $X = X_1 \oplus X_2$ is the Kato decomposition ($\dim X_1 < \infty, TX_1 \subset X_1, TX_2 \subset X_2, T_1 = T|X$ nilpotent and $T_2 = T|X_2$ s-regular), then $M = R^\infty(T_2) \subset X_2$ and $TM = T_2 M = M$. If $x = x_1 \oplus x_2$ satisfies $Tx \in M$, then $T_2 x_2 \in M$ so that $x_2 \in M$. Thus $x \in X_1 + M$ and $N(\tilde T) \subset X_1 + M$. Hence $\dim N(\tilde T) < \infty$.

Let $\pi : X \to X/M$ be the canonical projection. Since $M \subset R(T)$ and $R(\tilde T) = \{Tx + M : x \in X\} = \pi R(T)$, the range of $\tilde T$ is closed. Thus $\tilde T$ is upper semi-Fredholm.

Conversely, let M be a subspace of X with the required properties. We can prove that $R(T)$ is closed in exactly the same way as in Lemma 1.

Further, $M \subset R^\infty(T)$. If $Tx = 0$, then $\tilde T(x+M) = 0$, i.e. $\pi x \in N(\tilde T)$. Thus $N(T) \subset \pi^{-1}N(\tilde T) \subset M + F \subset R^\infty(T) + F$ for a finite-dimensional subspace $F \subset X$.

Theorem 6. *Let* $T, A \in B(X), TA = AT$, *and let* A *be a quasinilpotent. Then*

(1) $\sigma_\gamma(T+A) = \sigma_\gamma(T)$,
(2) $\sigma_{\gamma e}(T+A) = \sigma_{\gamma e}(T)$.

Proof. Let T be an essentially s-regular operator, and let A be a quasinilpotent commuting with T. Denote $M = R^\infty(T), T_1 = T|M$, and let $\tilde T : X/M \to X/M$ be the operator induced by T. Clearly $AM \subset M$ so that we can define operators $A_1 = A|M$ and $\tilde A : X/M \to X/M$ induced by A. Clearly $r(A_1) = \lim_{n\to\infty} \|A_1^n\|^{1/n} \le \lim_{n\to\infty} \|A^n\|^{1/n} = 0$ and $r(\tilde A) = \lim_{n\to\infty} \|\tilde A^n\|^{1/n} \le \lim_{n\to\infty} \|A^n\|^{1/n} = 0$ so that $\sigma(A_1) = \{0\}$ and $\sigma(\tilde A) = \{0\}$. Further $T_1 A_1 = A_1 T_1$ and $\tilde T \tilde A = \tilde A \tilde T$. Denote by

$$\sigma_\delta(T) = \{\lambda \in \mathbf{C} : T - \lambda \text{ is not onto}\},$$
$$\sigma_\pi(T) = \{\lambda \in \mathbf{C} : T - \lambda \text{ is not bounded below}\},$$
$$\sigma_{\pi e}(T) = \{\lambda \in \mathbf{C} : T - \lambda \text{ is not upper semi-Fredholm}\}$$

the defect spectrum, the approximate point spectrum and the essential approximate point spectrum, respectively.

By the spectral mapping property for these spectra we have

$$\sigma_\delta(T_1 + A_1) = \sigma_\delta(T),$$
$$\sigma_\pi(\tilde{T} + \tilde{A}) = \sigma_\pi(\tilde{T}),$$
$$\sigma_{\pi e}(\tilde{T} + \tilde{A}) = \sigma_{\pi e}(\tilde{T}).$$

Thus $0 \notin \sigma_\delta(T + A)$, i.e. $(T + A)M = M$. Similarly $0 \notin \sigma_{\pi e}(\tilde{T} + \tilde{A})$, i.e. $\tilde{T} + \tilde{A}$ is upper semi-Fredholm. By the previous lemma $T + A$ is essentially s-regular. This proves (2).

If T is s-regular and A a quasinilpotent commuting with T, then in the same way $(T + A)M = M$ and $\tilde{T} + \tilde{A}$ is bounded below. Hence $T + A$ is s-regular by Lemma 1.

Remark. Statement (1) for Hilbert space operators was proved in [10, Theorem 4.8]. The second statement gives a positive answer to Question 3 of [15].

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