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ON C^*-ALGEBRAS ASSOCIATED WITH LOCALLY COMPACT GROUPS

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Abstract. Let G be a locally compact group, and let G_d denote the same group G with the discrete topology. There are various C^*-algebras associated to G and G_d. We are concerned with the question of when these C^*-algebras are isomorphic. This is intimately related to amenability. The results can be reformulated in terms of Fourier and Fourier-Stieltjes algebras and of weak containment properties of unitary representations.

Introduction

Let G be a locally compact group and G_d the group G equipped with the discrete topology. Denote by λ_G and λ_{G_d} the left regular representation of G and G_d on $L^2(G)$ and $\ell^2(G_d)$, respectively. Let $C^*_\delta(G)$ and $C^*_r(G_d)$ be the C^*-subalgebras of $\mathcal{L}\big(L^2(G)\big)$ and $\mathcal{L}\big(\ell^2(G_d)\big)$ generated by $\lambda_G(G)$ and $\lambda_{G_d}(G_d)$, respectively.

When G is abelian, $C^*_r(G_d)$ via Fourier transform is isomorphic to the algebra $C(\widehat{G_d})$ of all continuous functions on the compact dual group $\widehat{G_d}$ of G_d. Similarly, $C^*_\delta(G)$ is isomorphic to the norm closed subalgebra of $L^\infty(\widehat{G})$ generated by the elements of G_d considered as functions on $\Gamma = \widehat{G}$ by means of the canonical isomorphism between G and $\widehat{\Gamma}$. By a classical result, $\widehat{G_d}$ is equal to the Bohr compactification $b\Gamma = \widehat{(\widehat{\Gamma})_d}$ of Γ. Moreover, the algebra $C(b\Gamma)$ of almost periodic functions on Γ considered as a subalgebra of $L^\infty(\Gamma)$ is generated by the set $G_d = \widehat{\Gamma}_d = \widehat{b\Gamma}$ of continuous characters on Γ. Hence, $C^*_\delta(G)$ is isomorphic to $C^*_r(G_d)$.

In this paper, we shall be concerned with possible extensions of this result to non-abelian groups.

Relationships between the two unital C^*-algebras $C^*_\delta(G)$ and $C^*_r(G_d)$ were studied by Dunkl and Ramirez [DuR] and by Bédos [Béd]. It was shown in [DuR], Theorem 2.5 (see also [Béd], Lemma 2) that λ_{G_d} extends to a (surjective) $*$-homomorphism

$$\Phi : C^*_\delta(G) \to C^*_r(G_d).$$

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This means that λ_G, when viewed as a unitary representation of G_d, weakly contains λ_{G_d} (concerning the notion of weak containment, we refer to [Dix], Chap. 18). For the convenience of the reader, we reproduce below (Proposition 1) a very short proof of this fact appearing in [BeV], Proposition 1.

A natural question is whether Φ is an isomorphism (that is, whether Φ is injective). This is easily settled in case G is amenable (see also [DuR], [Béd]): Φ is an isomorphism if and only if G_d is amenable. Indeed, in this case, by Hulanicki's theorem (see [Pat], Theorem 4.21 or [Pie], Theorem 8.9) the trivial one-dimensional representation 1_G is weakly contained in λ_G (as representations of G and, a fortiori, as representations of G_d). The claim now follows, as G_d is amenable if and only if 1_G is weakly contained in λ_{G_d}.

Our main result in this paper is a complete answer to the question raised above.

Theorem 1. *Let G be a locally compact group. Then $\Phi : C^*_\delta(G) \to C^*_r(G_d)$ is an isomorphism if and only if G contains an open subgroup H such that H_d is amenable.*

It is well-known that, for a connected Lie group G, G_d is amenable if and only if G is solvable. So, the following is an immediate consequence of Theorem 1.

Corollary 1. *Let G be a connected Lie group. Then Φ is an isomorphism if and only if G is solvable.*

In terms of weak containment, Theorem 1 states that λ_G is weakly contained in λ_{G_d} if and only if G contains an open subgroup H such that H_d is amenable. In fact, we shall prove the following stronger result:

Theorem 2. *Let G be a locally compact group. The following are equivalent:*

(i) λ_{G_d} *weakly contains* λ_G;
(ii) λ_{G_d} *weakly contains some continuous unitary representation of G, viewed as representation of G_d;*
(iii) *G contains an open subgroup H such that H_d is amenable.*

As is to be expected, the difficult part in the proof of this theorem is to show that (ii) implies (iii). This will require several steps, the main one being the case where G is a connected Lie group. The proof in this situation is based on the existence, for G non-solvable, of many non-abelian free subgroups of G and on estimates for the norm of convolution operators on free groups, similar to those appearing in [Kes], [Lei] and [AkO].

We now reformulate our results in terms of the Fourier and Fourier-Stieltjes algebras of G and G_d. Recall that the Fourier-Stieltjes algebra $B(G)$ of a locally compact group G is the linear span of all continuous positive definite functions on G. Recall also that $B(G)$ may be identified, in a natural way, with the dual of $C^*(G)$, the C^*-algebra of G. The Fourier algebra $A(G)$ of G is the closed subalgebra, generated by all functions in $B(G)$ with compact support. $A(G)$ may also be described as the set of all matrix coefficients of the regular representation λ_G of G. More details on $A(G)$ and $B(G)$ are to be found in [Eym] where these spaces are extensively studied.

Let $\overline{A(G_d)}^{w^*}$ denote the closure of $A(G_d)$ in $B(G_d)$, with respect to the weak* topology $\sigma\big(B(G_d), C^*(G_d)\big)$.

Then $\overline{A(G_d)}^{w^*}$ coincides with the space, denoted by $B_{\lambda_d}(G_d)$ in [Eym], of all matrix coefficients of the unitary representations of G_d which are the linear span of the positive definite functions associated with the unitary representations which are weakly contained in λ_{G_d} (see [Eym], (2.1) Proposition). Hence, the equality $\overline{A(G_d)}^{w^*} = B_{\lambda_d}(G_d)$ follows from [Eym], (1.2.1) Proposition.

As any continuous function in $B(G_d)$ actually lies in $B(G)$ ([Eym], (2.24) Corollaire 1), it is now clear that Theorem 2 may be reformulated as follows.

Theorem 2′. *Let G be a locally compact group. The following are equivalent:*

(i) $\overline{A(G_d)}^{w^*}$ *contains* $A(G)$;
(ii) $\overline{A(G_d)}^{w^*}$ *contains a non-zero continuous function on* G;
(iii) *G contains an open subgroup H such that H_d is amenable.*

This paper is organized as follows. In Section 1, we treat the case of connected Lie groups. The proof of Theorem 2 is then completed in Section 2. Section 3 contains some remarks about other C^*-algebras associated with G.

1. The connected Lie group case

In this section, we show that (ii) implies (iii) in Theorem 2 when G is a connected Lie group.

As mentioned above, we first reproduce the proof given in [BeV] for the existence of Φ.

Proposition 1. *λ_{G_d} is weakly contained in λ_G, for any locally compact group G.*

Proof. Since λ_{G_d} is cyclic, it suffices to show that δ_e, the Dirac function at the group unit e, is the pointwise limit of positive definite functions associated to λ_G. Let F be a finite subset of $G\backslash\{e\}$. Choose a neighbourhood K of e such that $gK \cap K = \emptyset$ for all $g \in F$. Set

$$\varphi(g) = \frac{1}{\mu(K)} \langle \lambda_G(g)\chi_K, \chi_K \rangle, \quad g \in G,$$

where χ_K is the characteristic function of K, and μ is a left Haar measure on G. Then φ is a positive definite function associated to λ_G such that $\varphi(g) = \delta_e(g)$ for all $g \in F \cup \{e\}$. □

We shall use the following estimate for the norm of convolution operators on free groups. This may easily be deduced from more general results appearing in [AkO], [Kes] and [Lei]. For the sake of completeness, we prefer to give an independent, short proof (compare also [BCH], Proof of Lemma 2.2).

Proposition 2. *Let Γ be a non-abelian free group on the generators x and y, and let λ denote the left regular representation of Γ on $\ell^2(\Gamma)$. Then*

$$\| \sum_{n=1}^{\infty} a_n \lambda(y^n x y^{-n}) \| \leq 2\|a\|_2$$

for all sequences $a = (a_n)_{n\in\mathbb{N}}$ in $\ell^2(\mathbb{N})$.

Proof. Let W_0 be the subset of Γ consisting of the words which do not begin with a nontrivial power of y. Let $W_n = y^n W_0$ for $n \in \mathbb{Z}$. Then $W_n \cap W_m = \emptyset$ for all $n \neq m$. For any $f, g \in \ell^2(\Gamma)$ and $n \in \mathbb{Z}$, the following holds

$$\begin{aligned} |\langle \lambda(y^n x y^{-n}) f, g \rangle| &= |\langle \lambda(y^n x y^{-n}) \chi_{W_n} f, g \rangle| + |\langle \lambda(y^n x y^{-n}) \chi_{\Gamma\backslash W_n} f, g \rangle| \\ &\leq \|\chi_{W_n} f\| \, \|g\| + \|f\| \, \|\chi_{y^n x y^{-n}\Gamma\backslash W_n} g\|, \end{aligned}$$

where χ_A denotes the characteristic function of $A \subseteq \Gamma$. Since $y^n x y^{-n}(\Gamma \backslash W_n) \subseteq W_n$, we get

$$|\langle \lambda(y^n x y^{-n}) f, g\rangle| \leq \|\chi_{W_n} f\| \, \|g\| + \|f\| \, \|\chi_{W_n} g\|$$

and therefore

$$\begin{aligned} |\langle \sum_{n=1}^{\infty} a_n \lambda(y^n x y^{-n}) f, g\rangle| &\leq \sum_{n=1}^{\infty} |a_n| (\|\chi_{W_n} f\| \, \|g\| + \|f\| \, \|\chi_{W_n} g\|) \\ &\leq 2\|a\|_2 \|f\| \, \|g\|, \end{aligned}$$

by the Cauchy-Schwarz inequality. □

Proposition 3. *Let G be a connected non-solvable Lie group. Then no continuous unitary representation of G, viewed as a representation of G_d, is weakly contained in λ_{G_d}.*

Proof. It is well-known that such a group contains a non-abelian free subgroup F on two generators a and b (see [Pat], Theorem 3.9). For any finite set of integers $i_1, \ldots, i_n \in \mathbb{Z} \setminus \{0\}$, let $p_{i_1 \ldots i_n} : G \to G$ denote the word function

$$p_{i_1 \ldots i_n}(x) = \begin{cases} a^{i_1} x^{i_2} \ldots a^{i_{n-1}} x^{i_n}, & \text{if} \quad n \quad \text{is even,} \\ a^{i_1} x^{i_2} \ldots x^{i_{n-1}} a^{i_n}, & \text{if} \quad n \quad \text{is odd.} \end{cases}$$

Then, the set

$$G_{i_1 \ldots i_n} = \{x \in G, \; p_{i_1 \ldots i_n}(x) \neq e\}$$

is open. It is also nonempty since $b \in G_{i_1 \ldots i_n}$. Moreover, because $p_{i_1 \ldots i_n}$ is an analytic function on G, $G_{i_1 \ldots i_n}$ is dense. So, by Baire's category theorem, the intersection

$$X = \bigcap \{G_{i_1 \ldots i_n} ; i_1, \ldots, i_n \in \mathbb{Z} \backslash \{0\}\}$$

is dense in G.

By the definition of X, for any $x \in X$, the subgroup Γ_x generated by a and x is a free group. Hence, by Proposition 2, for any $x \in X$ and $N \in \mathbb{N}$, we have the following estimate:

$$(*) \qquad \|\frac{1}{N} \sum_{n=1}^{N} \lambda_{\Gamma_x}(a^n x a^{-n})\| \leq \frac{2}{\sqrt{N}}$$

where λ_{Γ_x} denotes the regular representation of the discrete group Γ_x. But, since the restriction of λ_{G_d} to Γ_x is a multiple of λ_{Γ_x}, we may replace λ_{Γ_x} by λ_{G_d} in the above inequality $(*)$.

Now suppose, by contradiction, that there exists a continuous unitary representation π of G which is weakly contained in λ_{G_d}. Then, by $(*)$,

$$\|\frac{1}{N} \sum_{n=1}^{N} \pi(a^n x a^{-n})\| \leq \frac{2}{\sqrt{N}}$$

for any $x \in X$ and any $N \in \mathbb{N}$. Therefore, for any unit vector ξ in the Hilbert space of π, we have

$$(**) \qquad |\frac{1}{N} \sum_{n=1}^{N} \langle \pi(a^n x a^{-n})\xi, \xi \rangle| \leq \frac{2}{\sqrt{N}}$$

for all $x \in X$ and $N \in \mathbb{N}$. Since X is dense and π is (strongly) continuous, $(**)$ holds for any $x \in G$.

Taking $x = e$ and $N \geq 5$, we reach the contradiction

$$1 = \frac{1}{N} \sum_{n=1}^{N} \langle \pi(a^n e a^{-n})\xi, \xi \rangle \leq \frac{2}{\sqrt{5}}\ .$$

□

2. The general case

We now proceed with the proof of Theorem 2. This will require two results related to Proposition 3.

Proposition 4. *Let G be an amenable locally compact group. Assume that λ_{G_d} weakly contains some continuous unitary representation π of G. Then G_d is amenable.*

Proof. Let $\bar{\pi}$ denote the representation conjugate to π. Then the (inner) tensor product $\pi \otimes \bar{\pi}$ is weakly contained in $\lambda_{G_d} \otimes \bar{\lambda}_{G_d}$ and, hence, in λ_{G_d}, since $\lambda_{G_d} \otimes \bar{\lambda}_{G_d}$ is a multiple of λ_{G_d}.

On the other hand, because G is amenable, the trivial representation 1_G is weakly contained in $\pi \otimes \bar{\pi}$, by [Bek], Theorems 2.2 and 5.1. Hence, 1_G is weakly contained in λ_{G_d}. Therefore, G_d is amenable. □

Next, we extend Proposition 3 to all connected groups.

Proposition 5. *Let G be a connected locally compact group. Assume that λ_{G_d} weakly contains some continuous unitary representation π of G. Then G_d is amenable.*

Proof. By the structure theory for connected groups, G contains a compact normal subgroup K such that G/K is a Lie group (see [MoZ], p. 175).

We first claim that K_d is amenable. Indeed, since the restriction $\lambda_{G_d}|_K$ of λ_{G_d} to K is a multiple of λ_{K_d}, λ_{K_d} weakly contains $\pi|_K$. As K is amenable, the claim follows from Proposition 4.

Suppose, by contradiction, that $(G/K)_d$ is not amenable. Then, by the proof of Proposition 3, there exist an element $\dot{a} \in G/K$ and a dense subset $\dot{X}$ of G/K such that, for any $\dot{x} \in \dot{X}$, the subgroup generated by $\dot{a}$ and $\dot{x}$ is free.

Let $p : G \to G/K$ denote the canonical projection. Choose any $a \in G$ with $p(a) = \dot{a}$, and set $X = p^{-1}(\dot{X})$. Then, for any $x \in X$, the subgroup of G generated by a and x is free. As in the proof of Proposition 3, this, together with the fact that X is dense in G, yields a contradiction.

Therefore, $(G/K)_d$ and K_d are amenable. It follows that G_d is amenable. □

Proof of Theorem 2. That (i) implies (ii) is obvious. Suppose (ii) holds, that is, λ_{G_d} weakly contains some continuous representation π of G. Let G^0 denote the connected component of e in G. As $\lambda_{G^0_d}$ weakly contains $\pi|_{G^0}$, G^0_d is amenable by

Proposition 5. Since G/G^0 is totally disconnected, we may choose an open subgroup H of G containing G^0 such that H/G^0 is compact (see [HeR], Theorem 7.7). We claim that H_d is amenable. Indeed since H is amenable, the claim follows immediately from Proposition 4. This completes the proof that (ii) implies (iii).

Suppose (iii) holds, that is, G contains an open subgroup H such that H_d is amenable. Then λ_{H_d} weakly contains λ_H (in fact, λ_{H_d} weakly contains any unitary representation of H_d).

Now, since G/H is discrete, the induced representation $\mathrm{ind}_{H_d}^{G_d}\lambda_H$ is equivalent to $\mathrm{ind}_H^G\lambda_H = \lambda_G$. Therefore, by continuity of inducing, λ_G is weakly contained in $\mathrm{ind}_{H_d}^{G_d}\lambda_{H_d} = \lambda_{G_d}$. This shows that (i) holds and completes the proof of Theorem 2. □

3. Some remarks on other C^*-algebras associated with G

Let ω be the universal representation of the locally compact group G. Let $C^*(G_d)$ be the (maximal) C^*-algebra of G_d, and let $C^*_{\delta,\omega}(G)$ denote the C^*-algebra generated by all operators $\omega(x),\ x \in G$. There is an obvious surjective $*$-homomorphism

$$\Psi : C^*(G_d) \to C^*_{\delta,\omega}(G).$$

The main result in [BeV] may be reformulated as follows.

Theorem 3 ([BeV])**.** *Let G be a connected Lie group. Then Ψ is an isomorphism if and only if G is solvable.*

Remark 1. We do not know how to extend this result to other locally compact groups. A reasonable conjecture seems to be: Ψ is an isomorphism if and only if G contains an open subgroup H such that H_d is amenable (that is, by Theorem 1, if and only if Φ is an isomorphism).

Let $\Lambda : C^*_{\delta,\omega}(G) \to C^*_\delta(G)$ be the surjective $*$-homomorphism, defined in the natural way. The following result, for which we offer a simple proof, may also be deduced from [Béd, Theorem 1].

Theorem 4. *Let G be a locally compact group. Then Λ is an isomorphism if and only if G is amenable.*

Proof. If G is amenable, then λ_G is weakly equivalent to the universal representation ω (even as representations of G). This implies that Λ is an isomorphsim. Conversely, assume Λ is injective. Then 1_G is weakly contained in λ_G, where both representations are viewed as representations of G_d. That is, G has Reiter's weak property (P_2^*) (see [Pie], p. 56) which is known to characterise the amenability of G. □

Remark 2. We used in the above proof the surprising fact that Reiter's weak property (P_2^*) is equivalent to the stronger property (P_2) (saying that 1_G is weakly contained in λ_G, as representations of G) and, hence, to the amenability of G. Usually, the proof of this equivalence is a long and tedious one involving topological invariant means (compare [Pie], p. 56). It is worth mentioning that the argument used by Bédos in [Béd], Proof of Theorem 1, provides a quick and elegant proof for this equivalence. Indeed, assume 1_G is weakly contained in λ_G, as representations of G_d. Then 1_G defines a state φ on $C^*_\delta(G)$ such that $\varphi\big(\lambda_G(x)\big) = 1$ for all $x \in G$. Extend φ to a state $\widetilde{\varphi}$ on $\mathcal{L}\big(L^2(G)\big)$. Cauchy-Schwarz inequality shows that

$$\widetilde{\varphi}\big(\lambda_G(x)T\big) = \widetilde{\varphi}\big(T\lambda_G(x)\big)$$

for all $x \in G,\ T \in \mathcal{L}\big(L^2(G)\big)$.

Then, denoting by M_f the multiplication operator on $L^2(G)$ by $f \in L^\infty(G)$, one defines a mean m on $L^\infty(G)$ as follows:

$$m(f) = \widetilde{\varphi}(M_f), \quad \forall\, f \in L^\infty(G).$$

Since

$$M_{\lambda_G(x)f} = \lambda_G(x)M_f\lambda_G(x^{-1}), \quad \forall\, x \in G, \quad f \in L^\infty(G),$$

m is left invariant. This shows that G is amenable.

It should be observed that, using the notion of amenable representations as defined in [Bek], the above argument shows that λ_G is amenable.

The following corollary is also proved in [DuR], Proposition 3.2.

Corollary 2. *Let G be a locally compact group. Then*

$$\Phi \circ \Lambda : C^*_{\delta,\omega}(G) \to C^*_r(G_d)$$

is an isomorphism if and only if G_d is amenable.

Proof. If G_d is amenable, then, as shown earlier, Λ and Φ are isomorphisms.

Conversely, suppose $\Phi \circ \Lambda$ is an isomorphism. Since Λ is an isomorphism, by Theorem 4, G is amenable. As mentioned in the introduction, this implies that G_d is amenable, as Φ is an isomorphism. □

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