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A COUNTEREXAMPLE CONCERNING SMOOTH APPROXIMATION

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ABSTRACT. We answer a question of Smith, Stanoyevitch and Stegenga in the negative by constructing a simply connected planar domain Ω with no two-sided boundary points and for which every point on Ω^c is an m_2-limit point of Ω^c and such that $C^\infty(\overline{\Omega})$ is not dense in the Sobolev space $W^{k,p}(\Omega)$.

1. INTRODUCTION

Suppose $\Omega \subset \mathbb{R}^2$ is simply connected. Let $C^\infty(\overline{\Omega})$ denote the restriction to Ω of $C^\infty(\mathbb{R}^2)$ and let $W^{k,p}(\Omega)$ denote the Sobolev space of functions on Ω defined by

$$\|f\|_{W^{k,p}(\Omega)} = \|f\|_{L^p(\Omega)} + \sum_{|\alpha|\le k} \|D^\alpha f\|_{L^p(\Omega)} < \infty.$$

Let $\Omega^c = \mathbb{R}^2 \setminus \Omega$ denote the complement of Ω and let m_1, m_2 denote linear and two-dimensional Lebesgue measures. We say that a point $z \in E$ is an m_2 limit point of E if $m_2(E \cap B(z,r)) > 0$ for every disk centered at z. Also, $x \in \partial\Omega$ is called a two-sided boundary point of Ω if there is a $\delta(x) > 0$ so that for every $0 < \delta < \delta(x)$, $B(x,\delta) \cap \Omega$ has at least two components whose closures contain x.

Meyers and Serrin [1] proved that $C^\infty(\Omega)$ is always dense in $W^{k,p}(\Omega)$, but it is known that $C^\infty(\overline{\Omega})$ need not be. In their paper [2] Smith, Stanoyevitch and Stegenga prove several interesting theorems describing when $C^\infty(\overline{\Omega})$ is dense in $W^{k,p}(\Omega)$ and give several examples where it is not dense. Based on their results they asked the following.

Question. If Ω is a simply connected planar domain without two-sided boundary points and for which all points in Ω^c are m_2-limit points of Ω^c, then does it follow that $C^\infty(\overline{\Omega})$ is dense in $W^{k,p}(\Omega)$?

The purpose of this note is to show the answer is no.

2. THE CONSTRUCTION

We start by constructing a family of Cantor sets $E(t) \subset [0,1]$. These sets will depend continuously on $t \in [0,1]$ and will satisfy $m_1(E(t)) = t$. To begin, let

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$E_0(t) = [0,1]$ and let $I_0(t)$ be the open interval of length $\frac{1}{2}(1-t)$ centered at $\frac{1}{2}$. Then $E_1(t) = [0,1] \setminus I_0(t)$ consists of 2 closed intervals each of length $\frac{1}{2}(1-\frac{1}{2}(1-t)) = \frac{1}{4}(1+t)$. Let $I_2(t)$ be the union of the two open intervals, each of length $\frac{1}{8}(1-t)$ and concentric with the components of $E_1(t)$. Let $E_2(t) = E_1(t) \setminus I_2(t)$. Continue in this way, obtaining a sequence of nested compact sets $E_1(t) \supset E_2(t) \supset \cdots \supset E_n(t) \supset \ldots$ where $E_n(t) = E_{n-1}(t) \setminus I_n(t)$ consists of 2^n closed intervals, each of length $2^{-n}(1-(1-2^{-n})(1-t))$. The intersection of these sets is a Cantor set $E(t)$ of linear measure t.

Let F be any Cantor set of linear measure 1 in $[-1,1]$, say $F = 2E(1/2) - 1$, so that $m_1(F) = 1$. Let

$$K = \{(x,y) : -2 \le x \le 2, y \in E(\min(\frac{1}{2}, \mathrm{dist}(x,F)))\}.$$

See Figure 2.1. It may help to visualize the set if we note that K is homeomorphic to a Cantor set times an interval. The vertical cross sections of K are all sets of the form $E(t)$. For $x \in [-2, -\frac{3}{2}] \cup [\frac{3}{2}, 2]$ the cross section is $E(1/2)$ and for $x \in F$ it is $E(0)$. Since K has vertical cross section of positive linear measure for a open dense set of x's, K has positive area and every point of K is an m_2-limit of K.

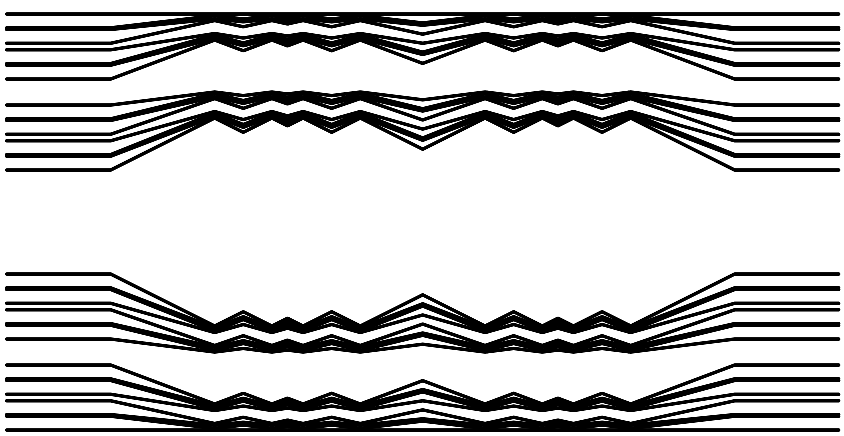

FIGURE 2.1. The set K.

The set K is not connected, and we add rectangles to make it so. More precisely, for n even let

$$R_n = [-2, -\frac{3}{2}] \times I_n(1/2),$$

and for n odd define

$$R_n = [\frac{3}{2}, 2] \times I_n(1/2).$$

Set $J = K \cup \bigcup_n R_n$. Each "horizontal tube" in the complement of K is now blocked by exactly one rectangle, so J is connected.

Let $\Omega = (-3,3) \times (0,3) \setminus J$. See Figure 2.2. It is easy to check that Ω is simply connected, has no two-sided boundary points and every point of Ω^c is an m_2-limit of Ω^c. Thus it only remains to show that $C^\infty(\overline{\Omega})$ is not dense in $W^{k,p}(\Omega)$. In fact, by Hölder's inequality we need only show it is not dense in $W^{1,1}(\Omega)$.

Let $\Omega_0 \subset \Omega$ be the component of $\Omega \cap \{(x,y) : -1 < x < 1\}$ which contains the point $(0,2)$. On Ω_0 let $f(x,y) = x$. On the two components Ω_+, Ω_- of $\Omega \setminus \Omega_0$ let

f be the constant ± 1, chosen to make f continuous. We claim that this function cannot be approximated in $W^{1,1}(\Omega)$ by elements of $C^\infty(\overline{\Omega})$.

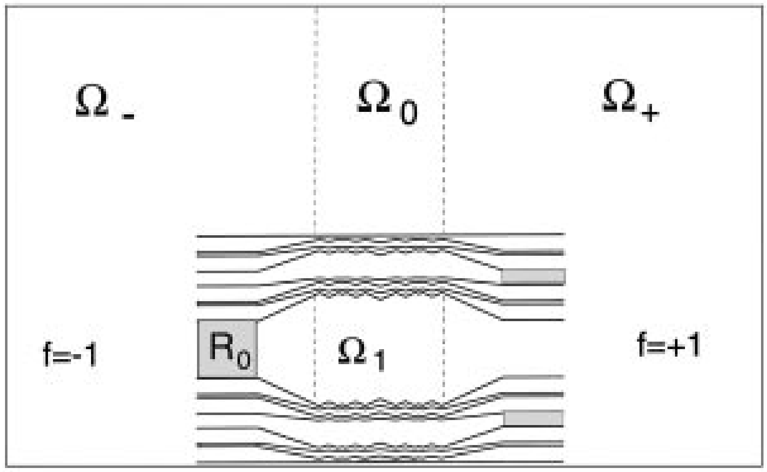

FIGURE 2.2. Ω and f.

Suppose $g \in C^\infty(\overline{\Omega})$. Let Ω_1 be the component of $\Omega \cap \{(x,y) : -1 \leq x \leq 1\}$ which contains the point $(0, \frac{1}{2})$. By the definition of $E(t)$, we see that $\Omega_1 \subset \Omega_+$ and

$$\{(x,y) : -1 < x < 1, \frac{3}{8} < y < \frac{5}{8}\} \subset \Omega_1 \subset \{(x,y) : -1 < x < 1, \frac{1}{4} \leq y \leq \frac{3}{4}\}.$$

Thus f is the constant 1 on Ω_1, so for any $\delta > 0$ there is an $\epsilon > 0$ such that

$$\int_\Omega |f - g| dxdy < \epsilon$$

implies $g > 1/2$ on a set $S \subset \Omega_1$ of measure $\geq (1-\delta) m_2(\Omega_1)$. If δ is small enough then the vertical projection of S must hit F in a set F' of measure at least $\frac{1}{2} m_1(F) = \frac{1}{2}$.

Write $F' = F_1 \cup F_2$ where

$$F_1 = \{x \in F' : g(x,y) \geq 0 \text{ for all } 0 \leq y \leq \frac{3}{4}\},$$

$$F_2 = \{x \in F' : g(x,y) \leq 0 \text{ for some } 0 \leq y \leq \frac{3}{4}\}.$$

One of these two sets must have measure greater than $\frac{1}{4} m_1(F) = \frac{1}{4}$.

First suppose $m(F_1) \geq \frac{1}{4}$. Then $|f - g| \geq 1$ on the set $(F_1 \times [0, 1/4]) \cap \Omega_-$. Thus

$$\|f - g\|_{W^{1,1}(\Omega)} \geq \int_\Omega |f - g| dxdy \geq m_2((F_1 \times [0, 1/4]) \cap \Omega_-) > 0,$$

independent of g.

On the other hand, suppose $m_1(F_2) \geq 1/4$. If $x \in F_2$, then g varies by at least $1/2$ on the segment $I_x = \{x\} \times [0, 3/4]$ so

$$\int_0^{3/4} |\frac{\partial g}{\partial y}(x,y)| dy \geq 1/2.$$

Since $\nabla f = 0$ on $\Omega \setminus \Omega_0$, we get

$$\|f-g\|_{W^{1,1}(\Omega)} \geq \int_{(F_2\times[0,3/4])\cap\Omega} |\frac{\partial g}{\partial y}(x,y)|dxdy.$$

Since $F_2 \subset F$, $x \in F_2$ implies $m_1(\Omega \cap I_x) = m_1(I_x)$; this is where we use the fact the vertical cross sections of K have zero length when $x \in F$. Hence

$$\int_{(F_2\times[0,3/4])\cap\Omega} |\frac{\partial g}{\partial y}(x,y)|dy = \int_{F_2\times[0,3/4]} |\frac{\partial g}{\partial y}(x,y)|dy,$$

and so

$$\|f-g\|_{W^{1,1}(\Omega)} \geq \int_{F_2\times[0,3/4]} |\frac{\partial g}{\partial y}(x,y)|dy \geq \frac{1}{2}m_1(F_2) \geq \frac{1}{8}.$$

Therefore in both cases we have shown that $\|f-g\|_{W^{1,1}(\Omega)}$ is bounded away from zero with an estimate independent of g. This proves that $C^\infty(\overline{\Omega})$ is not dense in $W^{1,1}(\Omega)$ and hence not dense in any $W^{k,p}(\Omega)$ for $k \geq 1$ or $1 \leq p < \infty$.

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