PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 124, Number 10, October 1996

A CLASS OF COMPLETE SECOND ORDER LINEAR DIFFERENTIAL EQUATIONS

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(Communicated by Palle E. T. Jorgensen)

ABSTRACT. This paper is concerned with a class of complete second order linear differential equations in a Banach space. We show the existence and uniqueness of classical solutions of

$$\text{(SE)} \qquad \begin{cases} u''(t) = A(t)u'(t) + B(t)u(t) + f(t) \text{ for } t \in [0,T] \\ u(0) = x \text{ and } u'(0) = y. \end{cases}$$

1. INTRODUCTION

In this paper we study complete second order linear differential equations

$$\text{(SE)} \qquad \begin{cases} u''(t) = A(t)u'(t) + B(t)u(t) + f(t) \text{ for } t \in [0,T] \\ u(0) = x \text{ and } u'(0) = y \end{cases}$$

in a Banach space X. Here $\{A(t) : t \in [0,T]\}$ is a family of closed linear operators in X with the common domain D of $A(t)$ and $\{B(t) : t \in [0,T]\}$ is a family of linear operators in X with $D \subset D(B(t))$ for $t \in [0,T]$.

In this case, as pointed out in [5], (SE) is closely related to the first order abstract Cauchy problem

$$(\text{CP};u_0, f) \qquad \begin{cases} u'(t) = A(t)u(t) + f(t) \text{ for } t \in [0,T] \\ u(0) = u_0. \end{cases}$$

Recently, Tanaka [10] has studied the problem $(\text{CP};u_0, f)$ above in the case where a family $\{A(t) : t \in [0,T]\}$ of closed linear operators in X satisfies all conditions which are usually referred to as the "hyperbolic" case except for the density of the common domain D of $A(t)$ and shown the existence and uniqueness result of classical solutions of $(\text{CP};u_0, f)$.

The purpose of this paper is to prove the existence and uniqueness of classical solutions of (SE) on the basis of the results on $(\text{CP};u_0, f)$ in [10]. Our result obtained extends Neubrander's one [7] on the special case of (SE) where $A(t) = A$ is independent of t and A is the infinitesimal generator of a (C_0)-semigroup on X (and so D is dense in X).

In what follows, for a Banach space X_i $(i = 1, 2)$, the associated norm is denoted by $\|\cdot\|_{X_i}$ and $L(X_1, X_2)$ denotes the Banach space of all bounded linear operators from X_1 into X_2. The norm in $L(X_1, X_2)$ is denoted by $\|\cdot\|_{X_1,X_2}$.

Received by the editors January 17, 1995 and, in revised form, April 3, 1995.
1991 *Mathematics Subject Classification.* Primary 42D05, 34G10.

2. The results on (CP;u_0, f)

In this section we recall the fundamental results obtained by Tanaka [10] on the problem

$$\text{(CP;}u_0, f) \qquad \begin{cases} u'(t) = A(t)u(t) + f(t) \text{ for } t \in [0,T] \\ u(0) = u_0 \end{cases}$$

and introduce the notion of a D-integral solution of (CP;u_0, f).

Here D is another Banach space with norm $\|\cdot\|_D$ which is continuously embedded in X. Assume the following three conditions on a family $\{A(t) : t \in [0,T]\}$ of closed linear operators in X.

(a1) $D(A(t)) = D$ is independent of t, and there exists a $c_0 > 0$ such that

$$c_0^{-1}\|y\|_D \le \|y\|_X + \|A(t)y\|_X \le c_0\|y\|_D \tag{2.1}$$

for $y \in D$ and $t \in [0,T]$.

(a2) There are constants $M \ge 1$ and $\omega \in (-\infty, \infty)$ such that

$$(\omega, \infty) \subset \rho(A(t)) \quad \text{for } t \in [0,T]$$

and

$$\left\| \prod_{j=1}^{k} (\lambda I - A(t_j))^{-1} \right\| \le M(\lambda - \omega)^{-k} \quad \text{for } \lambda > \omega \tag{2.2}$$

and every finite sequence $\{t_j\}_{j=1}^{k}$ with $0 \le t_1 \le t_2 \le \cdots \le t_k \le T$ and $k = 1, 2, \cdots$.

Then by (2.1) and (2.2) for $\lambda_0 > \omega_0 := \max(0, \omega)$ there exists a $c > 0$ such that

$$\|(A(t) - \lambda_0)^{-1}x\|_D \le c\|x\|_X \quad \text{for } x \in X. \tag{2.3}$$

(a3) For $y \in D$, $A(t)y$ is continuously differentiable in X.

First recall the results in [10] on the problem (CP;u_0, f).

By [10, Theorem 4.2] we obtain the following existence and uniqueness result of classical solutions of the problem (CP;u_0, f).

Theorem 2.1. *Suppose conditions* (a1)–(a3). *Let $f \in W^{1,1}([0,T] : X)$ and suppose that $u_0 \in D$ satisfies the compatibility condition that $A(0)u_0 + f(0) \in \overline{D}$. Then* (CP;$u_0, f$) *has a unique classical solution $u \in C([0,T] : D) \cap C^1([0,T] : X)$ given by*

$$u(t) = \lim_{\lambda \to 0+} \left(U_\lambda(t,0)u_0 + \int_0^{[t/\lambda]\lambda} U_\lambda(t,s)f(s)ds \right) \tag{2.4}$$

for $t \in [0,T]$, where $U_\lambda(t,s) = \prod_{i=[s/\lambda]+1}^{[t/\lambda]} (I - \lambda A(i\lambda))^{-1}$ for $0 \le s \le t \le T$.

Moreover u satisfies the integral equation

$$\begin{aligned} (A(t) - \lambda_0)u(t) + f(t) &= U(t,0)((A(0) - \lambda_0)u_0 + f(0)) \\ &\quad + \lim_{\lambda \to 0+} \int_0^{[t/\lambda]\lambda} U_\lambda(t,s)(\dot{A}(s)u(s) + f'(s) - \lambda_0 f(s))ds \end{aligned} \tag{2.5}$$

for $t \in [0,T]$, where $\lambda_0 > \omega_0$ and $U(t,0)z = \lim_{\lambda \to 0+} U_\lambda(t,0)z$ for $z \in \overline{D}$ and $t \in [0,T]$ (see (4.2) and (4.5) in [10]).

Remark 2.1. The combination of (2.4) and (2.5) gives

$$A(t)u(t) + f(t) = U(t,0)(A(0)u_0 + f(0)) + \lim_{\lambda\to 0+} \int_0^{[t/\lambda]\lambda} U_\lambda(t,s)(\dot{A}(s)u(s) + f'(s))ds \tag{2.6}$$

for $t \in [0,T]$.

One integrates (CP;u_0, f) formally and then an integration by parts yields

$$\begin{aligned} u(t) - u_0 &= \int_0^t A(s)u(s)ds + \int_0^t f(s)ds \\ &= A(t)\int_0^t u(s)ds - \int_0^t \dot{A}(s)\int_0^s u(r)drds + \int_0^t f(s)ds. \end{aligned}$$

This consideration leads us to the following definition.

Definition 2.1. Let $u_0 \in X$ and $f \in L^1([0,T]:X)$. A function $u:[0,T]\to X$ is called a *D-integral solution* of (CP;u_0, f) if $u \in C([0,T]:X)$, $\int_0^\cdot u(s)ds \in C([0,T]:D)$ and

$$u(t) - u_0 = A(t)\int_0^t u(s)ds - \int_0^t \dot{A}(s)\int_0^s u(r)drds + \int_0^t f(s)ds. \tag{2.7}$$

Remark 2.2. The notion of a D-integral solution of (CP;u_0, f) coincides with that of an integral solution introduced by Da Prato and Sinestrari [1] in the case where $A(t) = A$ is independent of t.

Let $u_0 \in \overline{D}$ and $f \in L^1([0,T]:X)$. Then it is shown in [10, Theorem 3.1] that the limit (2.4) exists uniformly in $t \in [0,T]$. Then we have

Theorem 2.2. *Suppose conditions* (a1)–(a3). *Let* $u_0 \in \overline{D}$ *and* $f \in L^1([0,T]:X)$. *A function* u *defined by* (2.4) *is a unique* D*-integral solution of* (CP;u_0, f). *Moreover we have the following estimates* :

$$\|u(t)\|_X \le C_1\left(\|u_0\|_X + \int_0^t \|f(s)\|_X ds\right); \tag{2.8}$$

$$\left\|\int_0^t u(s)ds\right\|_D \le C_1\left(\|u_0\|_X + \int_0^t \|f(s)\|_X ds\right) \tag{2.9}$$

for $t \in [0,T]$, *where* C_1 *is a constant independent of* u_0 *and* f.

Proof. Let $v_0(t) = 0$ on $t \in [0,T]$. Since $u_0 \in \overline{D}$, by Theorem 2.1 we can define $v_n \in C^1([0,T]:X) \cap C([0,T]:D)$ inductively by the unique classical solution of the problem

$$\begin{cases} v_n'(t) = A(t)v_n(t) + u_0 - \int_0^t \dot{A}(s)v_{n-1}(s)ds + \int_0^t f(s)ds \text{ for } t \in [0,T] \\ v_n(0) = 0 \end{cases}$$

for $n \ge 1$.

Then from (2.5) it follows that

$$\begin{aligned}(A(t)-\lambda_0)v_n(t)+u_0-\int_0^t \dot{A}(s)v_{n-1}(s)ds+\int_0^t f(s)ds \\ = U(t,0)u_0+\lim_{\lambda\to 0+}\int_0^{[t/\lambda]\lambda} U_\lambda(t,s)\Bigg[\dot{A}(s)v_n(s)-\dot{A}(s)v_{n-1}(s)+f(s) \\ -\lambda_0\left(u_0-\int_0^s \dot{A}(r)v_{n-1}(r)dr+\int_0^s f(r)dr\right)\Bigg]ds\end{aligned}$$

for $t \in [0,T]$. This together with (2.3) shows that there exists a constant $K > 0$ such that

$$\|v_{n+1}(t)-v_n(t)\|_D \le K\int_0^t (\|v_{n+1}(s)-v_n(s)\|_D+\|v_n(s)-v_{n-1}(s)\|_D)ds,$$

which implies by Gronwall's inequality that

$$\|v_{n+1}(t)-v_n(t)\|_D \le K\int_0^t e^{K(t-s)}\|v_n(s)-v_{n-1}(s)\|_D ds$$

for $t \in [0,T]$ and $n \ge 1$. By standard arguments we see that there exists an element $v \in C([0,T]:D)$ such that $\sup\{\|v_n(t)-v(t)\|_D : t \in [0,T]\}$ converges to zero as $n\to\infty$. Thus $v_n'(t)\left(= A(t)v_n(t)+u_0-\int_0^t \dot{A}(s)v_{n-1}(s)ds+\int_0^t f(s)ds\right)$ converges to $A(t)v(t)+u_0-\int_0^t \dot{A}(s)v(s)ds+\int_0^t f(s)ds$ uniformly in $t\in[0,T]$ as $n\to\infty$ and hence v' is a D-integral solution of (CP;u_0, f).

Let w be a D-integral solution of (CP;u_0, f) and let $w_1(t) = \int_0^t w(s)ds$. Then since $w_1'(t) = A(t)w_1(t)+u_0-\int_0^t \dot{A}(s)w_1(s)ds+\int_0^t f(s)ds$ and $w_1(0) = 0$, by (2.6) we have

$$\begin{aligned}w(t) &= w_1'(t) \\ &= U(t,0)u_0+\lim_{\lambda\to 0+}\int_0^{[t/\lambda]\lambda} U_\lambda(t,s)\{\dot{A}(s)w_1(s)+(-\dot{A}(s)w_1(s)+f(s))\}ds,\end{aligned}$$

which is equal to a function u defined by (2.4). This proves the uniqueness of D-integral solutions of (CP;u_0, f). This together with (2.4) shows (2.8).

Finally we shall show (2.9). By (2.1), (2.7) and (2.8) we get

$$\begin{aligned}c_0^{-1}\left\|\int_0^t u(s)ds\right\|_D &\le \left\|\int_0^t u(s)ds\right\|_X+\left\|A(t)\int_0^t u(s)ds\right\|_X \\ &\le \int_0^t \|u(s)\|_X ds+\left\|u(t)-u_0+\int_0^t \dot{A}(s)\int_0^s u(r)drds-\int_0^t f(s)ds\right\|_X \\ &\le C_2\left(\|u_0\|_X+\int_0^t \|f(s)\|_X ds\right)+\int_0^t \|\dot{A}(s)\|_{D,X}\left\|\int_0^s u(r)dr\right\|_D ds\end{aligned}$$

for $t \in [0,T]$, where C_2 is a constant independent of u_0 and f. Thus Gronwall's inequality gives the desired estimate (2.9). □

3. Complete second order linear differential equations

Now we turn to the complete second order linear differential equations

$$\text{(SE)} \qquad \begin{cases} u''(t) = A(t)u'(t) + B(t)u(t) + f(t) \text{ for } t \in [0, T] \\ u(0) = x \text{ and } u'(0) = y. \end{cases}$$

A family $\{A(t) : t \in [0, T]\}$ of closed linear operators in X is assumed to satisfy conditions (a1) – (a3) in Section 2. We assume the next two conditions on a family $\{B(t) : t \in [0, T]\}$ of linear operators in X :

(b1) For $t \in [0, T]$, $D \subset D(B(t))$ and $B(t) \in L(D, X)$.

(b2) For $y \in \cap_{0 \le t \le T} D(B(t))$, $B(t)y$ is continuously differentiable in X.

It should be noted that we do not assume the closedness of the operator $B(t)$ in X.

Definition 3.1. A function $u : [0, T] \to X$ is called a *D-classical solution* of (SE) if $u \in C^2([0, T] : X), u(t) \in D(B(t)), u'(t) \in D$, the functions $B(t)u(t), A(t)u'(t)$ are continuous in t and u satisfies (SE).

Let $x \in \cap_{0 \le t \le T} D(B(t))$. Consider the following first order differential equation:

$$\text{(FE)} \qquad \begin{cases} u'(t) = A(t)u(t) + B(t)x + B(t) \int_0^t u(s)ds + f(t) \text{ for } t \in [0, T] \\ u(0) = y. \end{cases}$$

We say that u is a *D-classical solution* of (FE) if $u \in C^1([0, T] : X) \cap C([0, T] : D)$ and u satisfies (FE).

Remark 3.1. If $u \in C([0, T] : D)$, then $\int_0^{\cdot} u(s)ds \in C([0, T] : D)$ by (a1) and the closedness of $A(0)$.

By assumptions (b1) and (b2) we obtain the following proposition immediately.

Proposition 3.1. *Suppose conditions* (a1)–(a3) *and* (b1)–(b2) *are fulfilled. A function $u : [0, T] \to X$ is a D-classical solution of* (FE) *if and only if $v(\cdot)$ defined by $v(t) = x + \int_0^t u(s)ds$ is a D-classical solution of* (SE).

Now we are in a position to state our main result in this paper.

Theorem 3.2. *Suppose conditions* (a1)–(a3) *and* (b1)–(b2) *are fulfilled. Let $f \in W^{1,1}([0, T] : X), x \in \cap_{0 \le t \le T} D(B(t))$ and $y \in D$. If the compatibility condition that $B(0)x + A(0)y + f(0) \in \overline{D}$ is satisfied, then* (SE) *has a unique D-classical solution v which satisfies*

$$\|v(t)\|_X \le C \left(\|x\|_X + \|y\|_X + \int_0^t \|B(s)x + f(s)\|_X ds \right) \tag{3.1}$$

for $t \in [0, T]$, where C is a constant independent of x, y and f.

Proof. By Proposition 3.1, it is sufficient to prove the existence and uniqueness of D-classical solutions of (FE).

Let $u_0(t) = y$ for $t \in [0, T]$. By Theorem 2.1 we can define

$$u_n \in C^1([0, T] : X) \cap C([0, T] : D)$$

inductively by the unique classical solution of the problem

$$\begin{cases} u_n'(t) = A(t)u_n(t) + B(t)x + B(t)\int_0^t u_{n-1}(s)ds + f(t) \text{ for } t \in [0,T] \\ u_n(0) = y \end{cases}$$

for $n \geq 1$.

Then by the same arguments used in proving Theorem 2.2 we see that u_n converges to an element u in $C([0,T]:D)$ as $n \to \infty$ in the topology of $C([0,T]:D)$ and that u is a D-classical solution of (FE).

To prove the uniqueness of D-classical solutions of (FE), let u_i $(i = 1,2)$ be D-classical solutions of (FE) and let $w = u_1 - u_2$. Then we have $w'(t) = A(t)w(t) + B(t)\int_0^t w(s)ds$ and $w(0) = 0$, and so by (2.5)

$$\begin{aligned}(A(t) - \lambda_0)w(t) &+ B(t)\int_0^t w(s)ds \\ &= \lim_{\lambda \to 0+} \int_0^{[t/\lambda]\lambda} U_\lambda(t,s)\Big[\dot{A}(s)w(s) \\ &\qquad + B(s)w(s) + \dot{B}(s)\int_0^s w(r)dr - \lambda_0 B(s)\int_0^s w(r)dr\Big]ds\end{aligned}$$

for $t \in [0,T]$. The estimation of this equality gives

$$\|w(t)\|_D \leq K'\int_0^t \|w(s)\|_D ds$$

for some constant $K' > 0$. By Gronwall's inequality we have $w = 0$.

Finally we shall show the estimate (3.1).

Since u is a D-integral solution of $\big(\mathrm{CP}; y, B(\cdot)x + B(\cdot)\int_0^{\cdot} u(s)ds + f(\cdot)\big)$, the estimate (2.9) gives

$$\begin{aligned}\left\|\int_0^t u(s)ds\right\|_D \leq& C_1\left(\|y\|_X + \int_0^t \left\|B(s)x + B(s)\int_0^s u(r)dr + f(s)\right\|_X ds\right) \\ \leq& C_1\left(\|y\|_X + \int_0^t \|B(s)x + f(s)\|_X ds\right) \\ &+ C_1\int_0^t \|B(s)\|_{D,X}\left\|\int_0^s u(r)dr\right\|_D ds,\end{aligned}$$

which implies by Gronwall's inequality that

$$\left\|\int_0^t u(s)ds\right\|_D \leq C_3\left(\|y\|_X + \int_0^t \|B(s)x + f(s)\|_X ds\right)$$

for some constant $C_3 > 0$. Combining this with (2.8) we obtain

$$\begin{aligned}\|u(t)\|_X &\leq C_1\left(\|y\|_X + \int_0^t \left\|B(s)x + B(s)\int_0^s u(r)dr + f(s)\right\|_X ds\right) \\ &\leq C_4\left(\|y\|_X + \int_0^t \|B(s)x + f(s)\|_X ds\right)\end{aligned}$$

for some constant $C_4 > 0$. Therefore Proposition 3.1 implies the desired estimate (3.1). □

The rest of this section is devoted to an application of Theorem 3.2 to the following hyperbolic partial differential equation :

$$\text{(P)} \begin{cases} u_{tt} + \alpha(t,x)u_{tx} + \beta(t,x)u_t = \gamma(t,x)u_x + \delta(t,x)u + f(t,x), \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad (t,x) \in [0,T] \times [0,1], \\ u(t,0) = u(t,1),\ t \in [0,T], u(0,x) = u_0(x),\ u_t(0,x) = u_1(x),\ x \in [0,1]. \end{cases}$$

We denote by X the Banach space $C[0,1]$ with norm $\|u\|_\infty = \sup\{|u(x)| : x \in [0,1]\}$ and by D the Banach space $\{u \in C^1[0,1] : u(0) = u(1)\}$ with norm $\|u\|_\infty + \|u'\|_\infty$. Define three families $\{A(t) : t \in [0,T]\}$, $\{B(t) : t \in [0,T]\}$ and $\{C(t) : t \in [0,T]\}$ of linear operators in X by

$$\begin{cases} D(A(t)) = D \\ (A(t)u)(x) = -\alpha(t,x)u'(x) \ \text{ for } u \in D, \end{cases}$$

$$\begin{cases} D(B(t)) = C^1[0,1] \\ (B(t)u)(x) = \gamma(t,x)u'(x) + \delta(t,x)u(x) \ \text{ for } u \in C^1[0,1], \text{ and} \end{cases}$$

$$\begin{cases} D(C(t)) = X \\ (C(t)u)(x) = -\beta(t,x)u(x) \ \text{ for } u \in X, \end{cases}$$

respectively.

If α is a positive function of class C^1, then the family $\{A(t) : t \in [0,T]\}$ of closed linear operators satisfies three conditions (a1) – (a3) (see [2, Theorem 6.1]). Also, it is well-known that the stability condition is preserved under the perturbation of a uniformly bounded family of bounded linear operators on X (see [4] or [9, Theorem 5.2.3]). Therefore, if β is of class C^1, the family $\{A(t) + C(t) : t \in [0,T]\}$ of closed linear operators in X satisfies conditions (a1) – (a3) with $A(t)$ replaced by $A(t) + C(t)$. It is easy to see that conditions (b1) – (b2) are satisfied if γ and δ are class C^1. Since $\overline{D} = \{u \in C[0,1] : u(0) = u(1)\}$, Theorem 3.2 asserts that if $f \in W^{1,1}([0,T] : C[0,1]), u_0, u_1 \in C^1[0,1], u_1(0) = u_1(1)$ and the compatibility condition that $-\alpha(0,0)u_1'(0) - \beta(0,0)u_1(0) + \gamma(0,0)u_0'(0) + \delta(0,0)u_0(0) + f(0,0) = -\alpha(0,1)u_1'(1) - \beta(0,1)u_1(1) + \gamma(0,1)u_0'(1) + \delta(0,1)u_0(1) + f(0,1)$ is satisfied, then the problem (P) has a unique solution $u \in C^2([0,T] : C[0,1]) \cap C([0,T] : C^1[0,1])$ which satisfies the estimate :

$$\sup_{x\in[0,1]} |u(t,x)| \le C\Bigg(\sup_{x\in[0,1]} |u_0(x)| + \sup_{x\in[0,1]} |u_1(x)| + \int_0^t \sup_{x\in[0,1]} |\gamma(s,x)u_0'(x) + \delta(s,x)u_0(x) + f(s,x)|ds \Bigg)$$

for $t \in [0,T]$, where C is a constant independent of u_0, u_1 and f.

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