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NORMAL SUBGROUPS OF $PSL_2(Z[\sqrt{-3}])$

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ABSTRACT. We classify the normal subgroups of $PSL_2(Z[\sqrt{-3}])$ of index less than 960; they are all congruence subgroups.

I. INTRODUCTION

In Fine's monograph [F], the group $PSL_2(Z[\sqrt{-3}])$ appears exceptional among the Euclidean Bianchi groups since it is not amenable to the standard techniques of combinatorial group theory, HNN extensions and free products with amalgamation. In [A], we showed that Γ has a particularly interesting triangle of finite groups structure coming from its action on a contractible simplicial complex; now, from our Corollary 1 below it follows that Γ has no non-trivial action on a tree and therefore, *a fortiori*, cannot be expressed as an amalgamation or HNN extension.

We now review the main results on the structure of Γ. Let $R = Z[\sqrt{-3}]$ be the ring of integers in the field $Q(\sqrt{-3})$. According to the results of [A], the group $\Gamma = PSL_2(R)$ acts on a contractible simplicial complex $\mathcal{X}$ with a fundamental domain which is a single 2-simplex. The edge stabilizers $\langle a\rangle, \langle b\rangle, \langle c\rangle$ and vertex stabilizers A, B, C are as indicated: $a = \begin{pmatrix} 0 & 1 \\ -1 & 1 \end{pmatrix}$, $b = \begin{pmatrix} 0 & \omega^2 \\ \omega & 0 \end{pmatrix}$, $c = \begin{pmatrix} 0 & \omega \\ \omega^2 & 0 \end{pmatrix}$ where $\omega = \frac{1+\sqrt{-3}}{2}$ and

$$A = \langle b, c | b^2 = c^2 = (bc)^3 = 1\rangle,$$
$$B = \langle a, c | a^3 = c^2 = (a^{-1}c)^3 = 1\rangle,$$
$$C = \langle a, b | a^3 = b^2 = (ab)^3 = 1\rangle.$$

As shown in [A], it follows that the presentation for Γ is

$$\langle a, b, c | a^3 = b^2 = c^2 = (ab)^3 = (a^{-1}c)^3 = (bc)^3 = 1\rangle.$$

We easily see that the group Γ with this presentation admits a triangle of groups structure, according to Gersten and Stallings, with the stabilizer groups as given. More importantly, it is a non-positively curved triangle with all three angles $\frac{\pi}{3}$; this results immediately from the three relations $(bc)^3 = 1$, $(a^{-1}c)^3 = 1$, $(ab)^3 = 1$ and the structure of the groups A, B, C. Thus, from the Bounded Subgroup Theorem [St], any subgroup of finite order is conjugate to a subgroup of A, B, or C. Alternatively, as Bridson showed in his thesis, at the author's instigation, the

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complex $\mathcal{X}$ is non-positively curved and from the Fixed Point Theorem, the finite subgroups of Γ are determined [B, Sec. 42]. The triangle of groups structure is used in an essential way in Proposition 2 below.

By using this triangle structure, together with the solvable series, and elementary results about the simple groups of small order, we can deduce information on normal subgroups along the lines used for the Picard group [F-N]. Specifically, we determine the congruence kernels of small level. The (congruence) kernel of the natural homomorphism $PSL_2(R) \to PSL_2(R/(\alpha))$, $\alpha \in R$, is denoted $\Gamma(\alpha)$. A congruence subgroup is a subgroup containing some congruence kernel $\Gamma(\alpha)$. We shall denote, as usual, the terms of the derived series of Γ by $\Gamma^{(i)}$, $i \geq 0$.

Our main result is the following.

Theorem. *A non-trivial normal subgroup of* Γ *of index less than* 960 *is one of the following congruence subgroups:* $\Gamma^{(1)}$ *of index* 3*;* $\Gamma^{(2)} = \Gamma(1+\omega)$ *of index* 12*;* $\Gamma(2)$ *of index* 60*;* $\Gamma(2+\omega)$ *or* $\Gamma(3-\omega)$ *of index* 168*;* $\Gamma^{(1)} \cap \Gamma(2)$ *of index* 180*;* $\Gamma^{(3)} = \Gamma(3)$ *of index* 324*;* $\Gamma^{(1)} \cap \Gamma(2+\omega)$ *or* $\Gamma^{(1)} \cap \Gamma(3-\omega)$ *of index* 504*.*

II. Triangle structure and solvable quotients

A group G has property FA (of Serre) if every action (without inversions) on a tree has a fixed point. This implies that the group cannot be decomposed via HNN extensions, or free products with amalgamation. A group G has property T (of Kazhdan), roughly speaking, if its non-trivial irreducible unitary representations are uniformly bounded away from the identity. Property T implies that G has finite abelianization. It is known that property T implies property FA [A1].

Proposition 1. *Suppose that* G *is generated by three subgroups* U, V, W *so that the subgroups generated by any two,* $X = \langle V, W\rangle$*,* $Y = \langle U, W\rangle$ *and* $Z = \langle U, V\rangle$ *have property FA then* G *has property FA.*

Proof. Suppose that G acts on the tree T; X, Y and Z fix the vertices x, y and z of T. Consider a common point P in $yz \cap xz \cap xy$. The segment yz is fixed by U, xz is fixed by V and xy is fixed by W. Thus the common point P is fixed by G. □

Corollary 1. Γ *has property FA.*

Proof. The edge stabilizers $\langle a\rangle, \langle b\rangle, \langle c\rangle$ are finite, as well as the subgroups generated by any two of these, $\langle a, b\rangle, \langle b, c\rangle, \langle a, c\rangle$; hence they all have property FA and the corollary now follows immediately from Proposition 1.

We shall consider the finite index normal subgroups of Γ. Using the triangle of groups structure we can limit the possible normal subgroups. A homomorphism of Γ to a quotient G is determined by its restriction to the subgroups A, B, C or the elements a, b, c. The homomorphic images of A are either $\{1\}$, Z_2, or A. The homomorphic images of B (resp. C) are $\{1\}$, Z_3, or B (resp. C).

We consider the finite solvable quotients of Γ. If G is a finite quotient of Γ, then the induced homomorphism gives a surjection also from $\Gamma/\Gamma^{(i)}$ onto $G/G^{(i)}$. □

Theorem 1. *The finite solvable quotients of* Γ *are either the abelianization* Z_3*, the congruence quotient* $\Gamma/\Gamma^{(2)} = PSL_2(R/(1+\omega))$ *of order* 12*, or the congruence quotient* $\Gamma/\Gamma^{(3)} = PSL_2(R/(3))$ *of order* 324*. Any other finite solvable quotient has order* $324k$*,* $k \geq 3$*.*

The proof will follow from the next few propositions.

Lemma 1. *The derived series quotients are as follows:* $\Gamma/\Gamma^{(1)}$ *has order* 3, $\Gamma^{(1)}/\Gamma^{(2)}$ *is of order* 4 *and exponent* 2, $\Gamma^{(2)}/\Gamma^{(3)}$ *is of order* 27 *and exponent* 3.

Proof. An easy calculation shows that $\Gamma/\Gamma^{(1)} = Z_3$ and thus $\Gamma^{(1)} = \langle b, c, aba^{-1}, aca^{-1}\rangle$. An easy Reidemeister rewriting yields the presentation $\Gamma^{(1)} = \langle p, q, r, s | p^2 = q^2 = r^2 = s^2 = (qp)^2 = (rs)^2 = (pr)^3 = (qs)^3 = (qprs)^3 = 1\rangle$. It follows that the abelianization of $\Gamma^{(1)}$ is or order 4 and exponent 2 and hence $\Gamma^{(2)} = \langle bc, abca^{-1}, a^{-1}bca\rangle$. Another Reidemeister rewriting yields the presentation $\Gamma^{(2)} = \langle u, v, w | u^3 = v^3 = w^3 = (uv^{-1})^3 = (uw^{-1})^3 = (vw^{-1})^3 = 1\rangle$. The abelianization of $\Gamma^{(2)}$ is immediately seen to be a group of order 27 with exponent 3. □

Proposition 2. *The non-trivial homomorphic images of* Γ *are either isomorphic to* Z_3*, or to* A_4*, or have a triangle amalgam structure, injective on* $A \cup B \cup C$*. In the first case, the kernel is* $\Gamma^{(1)}$*, normally generated by* b *and* c*. In the second case, the kernel is* $\Gamma^{(2)}$*, normally generated by* bc *and is the congruence kernel* $\Gamma(1+\omega)$*. These two kernels have torsion. A triangle amalgam, injective on* $A \cup B \cup C$*, contains a subgroup isomorphic to* A_4 *and hence* (*if finite*) *has index* $12k, k > 2$*. The corresponding kernel is torsion-free.*

Proof. If the homomorphism is faithful on A, B, C the kernel is torsion-free, since any element of finite order is conjugate to an element of A, B, or C [St]. The kernel has index divisible by 12, since $B \cong A_4$ has order 12 and is isomorphic to a subgroup of the quotient. Now, we shall consider several cases. In the first case, suppose that the images of b and c are both trivial, then the image is generated by a and is therefore of order 3 and from Lemma 1 the kernel is $\Gamma^{(1)}$ or the image is trivial. Next, observe from the presentation of A that if one of b or c is mapped trivially, then all of A is mapped trivially and hence this reduces to the first case.

Now, suppose that the homomorphism is injective on A, then the image of a is also non-trivial (from the presentation of B) and hence the map is faithful on B and C also, else the image of B, say, would have order 3 and thus c would necessarily be trivial. Therefore, if the map is not faithful on A, B, C we must have that the images of b and c are nontrivial and the image of A is Z_2. Hence, the image of b and c are equal and therefore the images of B and C are equal; hence the image is the faithful image of B which is of order 12, isomorphic to A_4. This determines a unique normal subgroup of index 12, hence it is $\Gamma^{(2)}$. Since $R/(1+\omega)$ is a field with three elements, $PSL_2(R/(1+\omega))$ has order 12; thus, the congruence kernel for $1+\omega$ gives this unique normal subgroup of index 12.

Now, suppose the homomorphism is faithful on A, B, C, and the image has more than 12 elements. Then, it is injective on $A \cup B \cup C$. For otherwise, there is a coincidence in the image of B and C, A and C, or A and B. However in the first case this implies the image of $B \cup C$ is B, and in the others that either $A \subseteq B$ or $A \subseteq C$; then the image has exactly 12 elements. Finally, if G is a finite quotient of order 24, then G is isomorphic to $A \cup B \cup C$ which also has 24 elements. The subgroup B is normal of index 2. This contradicts $\Gamma/\Gamma^{(1)} \cong Z_3$. □

Corollary 2. *Any homomorphism of* Γ *to* $SL_2(F)$*, for any field* F *of odd characteristic, is either trivial or has image of order three.*

Proof. If F is algebraically closed of odd characteristic, then it follows easily from Jordan form that $SL_2(F)$ has a unique (central) element of order 2. However, any quotient with more than three elements contains a group isomorphic to A_4

by Proposition 2. This has three elements of order 2. Thus the image of Γ via a homomorphism to $SL_2(F)$ has image of order dividing 3. □

Lemma 2. $\Gamma/\Gamma^{(3)} = PSL_2(R/(3))$ *and this has no quotients of order* 36 *or* 108.

Proof. We have that $\Gamma^{(2)}/\Gamma^{(3)}$ is abelian and has order 27 and that $\Gamma/\Gamma^{(1)}$ is isomorphic to $PSL_2(R/((1+\omega)))$. We claim that in fact $\Gamma/\Gamma^{(3)}$ must be $PSL_2(R/(3))$. Since $(3) = (1+\omega)^2$, we have the natural homomorphism

$$PSL_2(R/3) \to PSL_2(R/(1+\omega)).$$

The kernel of this can be identified with the group of matrices of the form $1 + X$ where X is a matrix in $M_2((1+\omega)/(3))$ of trace 0. From this description we see that $PSL_2(R/(3))$ has the same solvable series as Γ up to step 3 and therefore $\Gamma^{(3)} = \Gamma(3)$. Moreover, since the natural homomorphism $\Gamma \to PSL_2(R/(1+\omega))$ splits, $PSL_2(R/3)$ is also a split extension. We can identify this extension then as having quotient $PSL_2(Z_3)$ and the kernel is the additive group of matrices of trace zero, $M_2^0(Z_3)$, with the action given by conjugation. Now, an easy exercise shows that $M_2^0(Z_3)$ is irreducible as a $PSL_2(Z_3)$ module, and hence it follows that there cannot be any quotients of $\Gamma/\Gamma^{(3)}$ of order 36 or 108. □

Proposition 3. *There are no quotients of* Γ *of order* $12p^k$ *for any prime* $p \neq 2, 3, 5$. *There are no solvable quotients of* Γ *of order less than* 324 *other than the derived series quotients. Hence, all other solvable quotients have order divisible by* 324*; there are no solvable quotients of order* 648.

Proof. If G has order $12p^k$, $p \neq 2, 3, 5, 11$, then the Sylow p-subgroup must be normal and thus G is solvable; it follows that $G/G^{(3)}$ has order $12p^j$ for $1 \leq j \leq k$, but this is not a divisor of 324. If G has order $12 \cdot 11^k$ and a Sylow 11-subgroup P is not normal, the it has 12 conjugates; we obtain a homomorphism $G \to S_{12}$ with kernel contained in P, and so the image is of order 132, hence solvable of length ≤ 3, but not a divisor of 324.

For any solvable finite quotient G of Γ, $G/G^{(1)} \cong Z_3$; also $G/G^{(2)} \cong A_4$ since $\Gamma/\Gamma^{(2)}$ has order 12 and this has a unique proper quotient of order 3. Now $G^{(2)}/G^{(3)}$ can only have order 27, since $\Gamma^{(2)}/\Gamma^{(3)}$ has order 27 and $\Gamma/\Gamma^{(3)}$ has no quotients of order 36 or 108.

If $G = \Gamma/N$ is a quotient of order 648, then it must be a central extension of $\Gamma/\Gamma^{(3)}$ by $\Gamma^{(3)}/N$ of order 2. However the normal subgroup $\Gamma^{(2)}/N$ is then a direct product of its (characteristic) 2-Sylow subgroup and 3-Sylow subgroup which thus gives rise to a normal subgroup of index 24 in Γ which is impossible by Proposition 2.

The proof of Theorem 1 is now immediate.

In Γ, there are the two unipotent elements $x = abcb$ and $y = bc(abcb)cb$ which commute and are conjugate via bc. Hence, in any finite quotient these have the same finite order. We shall denote the image of $\langle x, y\rangle$ by A. The normal subgroup generated by a set S is denoted $\langle\langle S\rangle\rangle$. □

Corollary 3. $\Gamma^{(3)} = \langle\langle (abcb)^3 \rangle\rangle$.

Proof. A coset enumeration shows that $\Gamma/\langle\langle (abcb)^3\rangle\rangle$ has order 324. From Lemma 2, $\Gamma/\Gamma^{(3)} = PSL_2(R/(3))$ and the unipotent element x has order 3 in this quotient; hence $\langle\langle (abcb)^3\rangle\rangle \subseteq \Gamma(3) = \Gamma^{(3)}$ and all have index 324. □

Corollary 4. $\Gamma^{(3)}/\Gamma^{(4)}$ *is infinite and hence* Γ *does not have property* T.

Proof. Since the abelianization of a group with property T is finite and subgroups of finite index in a group with property T also have property T, it follows that all the derived series quotients must be finite. However, according to a result of Serre ([S], cf. 3.5, Corollaire 3) the abelianization of the torsion-free $\Gamma(3) = \Gamma^{(3)}$ has non-zero torsion-free rank coming from the cusps. Hence Γ does not have property T. □

Remark. More generally, for $G = PSL_2(\mathcal{O})$, an imaginary quadratic ring of integers $\mathcal{O}$, $G/G^{(1)}$ is infinite, except for the ring of integers $Z[\sqrt{-1}]$ or $Z[\sqrt{-3}]$. This follows from Serre's result quoted above. The case of $Z[\sqrt{-1}]$ is treated in [F-N].

III. Non-solvable quotients

We shall now examine the non-solvable quotients of order less than 960. We make further use of the two commuting unipotent elements $x = abcb$ and $y = bc(abcb)cb$.

Proposition 4. *In a finite quotient of* Γ, G, *if* x *and* y *have order* m *and generate a cyclic group, then either* $y = x$ *and* G *has order dividing* 3, *or* $3|\phi(m)$.

Proof. In the image, the commuting unipotents x and $y = uxu^{-1}$ are conjugate via $u = bc$ (of order dividing 3) and hence have the same order say m. If the group A is cyclic, then it is of order m, generated by x say, and $y = x^k$ for $0 \leq k < m$. So $uxu^{-1} = x^k$, and repeated conjugation by u yields that $k^3 = 1 \mod m$. In the first case $k = 1$, then $c = baba^{-1} = a^{-1}ba$ so that G is a quotient of the group $\langle a, b\rangle$. We also have the relations $(aba)^3 = 1$ or $aba^{-1} = ba^{-1}ba$ and the relations $(ba^{-1}ba)^3 = 1$. Simplifying these leads to $b = 1$ and thus G has order dividing 3. If $k \neq 1$ it then follows immediately that $3|\phi(m)$. □

Lemma 3. *The finite quotient of* $\Gamma/\langle\langle x^2\rangle\rangle$ *has order* 60.

Proof. A coset enumeration shows that $\Gamma/\langle\langle(abcb)^2\rangle\rangle$ has order 60. □

Theorem 2. $\Gamma(2)$ *is the unique normal subgroup of index* 60.

Proof. If G is a finite quotient of order 60, it is either perfect or solvable. The latter case is impossible. Hence the group is the simple group of order 60. The simple group of order 60 has elements only of orders 2, 3, or 5. From Proposition 4 it follows that x and y generate a non-cyclic group A of order 4. However, from Lemma 3, the group $\Gamma/\langle\langle x^2\rangle\rangle$ has order 60; since $\langle\langle(abcb)^2\rangle\rangle \subseteq \Gamma(2)$, and $PSL_2(R/(2)) \cong PSL_2(GF(4))$ has order 60, these normal subgroups are equal.

In Proposition 4, for the case $m = 7$, there are the two solutions $k \in \{2, 4\}$ for $y = x^k$. An easy coset enumeration shows that the groups $\Gamma/\langle\langle y = x^2\rangle\rangle$ and $\Gamma/\langle\langle y^{-1} = x^3\rangle\rangle$ have order 168. Let $N_1 = \langle\langle y^{-1}x^2\rangle\rangle$ and $N_2 = \langle\langle yx^3\rangle\rangle$ be the corresponding normal subgroups of Γ of index 168. □

Theorem 3. Γ *has exactly two different normal subgroups* $N_1 = \Gamma(2 + \omega)$, $N_2 = \Gamma(3 - \omega)$ *of index* 168 *with isomorphic quotients, the simple group* $PSL_2(Z_7)$.

Proof. We can and do assume that the quotient group is non-solvable and hence is the simple group of order 168. This group has non-trivial elements of orders 2, 3, 4 or 7. By Proposition 4, x and y cannot generate a cyclic group of order $m \in \{2, 3, 4\}$ since 3 does not divide $\phi(m)$. Also x, y cannot generate a non-cyclic group of order 4 by Lemma 3. Thus we must have x, y of order 7 and thus $y = x^2$ or $y^{-1} = x^3$. The remarks preceding the theorem give us the two normal subgroups N_1, N_2. The relation $y = x^2$ implies that N_1 is the congruence kernel for $(2+\omega)$. The relation $y^{-1} = x^3$ implies that N_2 is the congruence kernel for $(3-\omega) = (2+\omega^{-1})$. To see that these normal subgroups are different we check that Γ/N_1N_2 is the trivial group. To see this observe that if $y = x^2 = x^{-3}$, then $x^5 = 1$; therefore, since $x^7 = 1$, $x = 1$. Hence $a = bcb$ and it follows that the element a has order 2, but this implies $a = 1$ and then also $b = c = 1$. □

Corollary 5. *Γ has no finite quotients isomorphic to $PSL_2(F)$, for $|F| = 8, 9, 11$.*

Proof. For $|F| = 8, 9, 11$, A can be non-cyclic of order 4. For $|F| = 9$, A can be non-cyclic of order 9. A non-cyclic A of order 4 is impossible by Lemma 3. The case of non-cyclic A of order 9 is impossible by Corollary 3. For $|F| = 8$, A cannot be cyclic of order $m = 2, 3, 9$ by Proposition 4, or cyclic of order 7 by the remarks preceding Theorem 3. For $|F| = 9$, A cannot be cyclic of order $m = 2, 3, 4, 5, 6$ by Proposition 4. In case $|F| = 11$, it follows from Proposition 4, A cannot be cyclic of order 2, 3, 5, 6, 11. □

Theorem 4. *A non-trivial normal subgroup of Γ of index less than 960, with non-solvable quotient, is one of the congruence subgroups $\Gamma(2)$, $\Gamma(2+\omega)$, $\Gamma(3-\omega)$, $\Gamma^{(1)} \cap \Gamma(2)$, $\Gamma^{(1)} \cap \Gamma(2+\omega)$ or $\Gamma^{(1)} \cap \Gamma(3-\omega)$.*

Proof. For the first case, assume N is a normal subgroup of finite index of Γ contained in normal subgroup M so that $K = M/N$ is simple and $S = \Gamma/M$ is solvable of order less than 324; then S is Z_3 or A_4. We obtain an induced homomorphism, via conjugation, from S to $\mathrm{Out}(K)$. However, for the simple groups of order less than 960 this map must be trivial. Let $G = \Gamma/N$, then the induced map via conjugation is a surjection, $G \to K$ with kernel $C_G(K)$, since K has a trivial center. Thus, $C_G(K) \cap K = 1$ so that G is a direct product.

In the second case, assume N is a normal subgroup of finite index of Γ contained in normal subgroup M so that $K = M/N$ is solvable of order less than 16 and $S = \Gamma/M$ is simple of order less than 960. Let $G = \Gamma/N$. Since S is assumed simple, and the restrictions on the order of K, the induced homomorphism, via conjugation, from S to $\mathrm{Out}(K)$ is trivial. As above, we obtain a surjective homomorphism, induced via conjugation, $G \to K/Z(K)$ with kernel $C_G(K)$. If K is abelian, then $C_G(K) = G$ so that G is a central extension of a simple group of order less than 960. Otherwise, $K/Z(K)$ is solvable, and the only homomorphism possible with the given restrictions on K, is from G onto $K = A_4$, but then G has less than 60 elements. Now the only central extensions of a simple group of order less than 960 are either the simple groups of order less than 960, trivial central extensions, or the non-trivial central extensions $SL_2(Z_5)$, $SL_2(Z_7)$, $SL_2(GF(9))$. The latter cannot occur by Corollary 2.

Since the first case analysis yields only direct products, it then follows that any finite index normal subgroup falls into one of the two cases and thus we have that the finite quotients are those already determined or direct products. Thus the finite index normal subgroups are those given in the statement of the Theorem. □

Question. Suppose that we have a surjection $PSL_2(R) \to PSL_2(F)$, F, a field with $|F| > 5$. Is it then the case that there is a surjection $R \to F$? Is this also true more generally if we replace R by any other imaginary quadratic rings of integers? It follows from the positive solution to the congruence subgroup problem that this question has an affirmative answer for rings of integers other than imaginary quadratics.

References

[A] Roger C. Alperin, *Homology of* $PSL_2(Z[\omega])$, Comment. Math. Helv. **55** (1980), 364–377. MR **82f:**20070

[A1] ———, *Locally compact groups acting on trees and property* T, Monatsh. Math. **93** (1982), 261–265. MR **83m:**22009

[B] Martin R. Bridson, *Geodesics and curvature in metric simplicial complexes*, Group Theory from a Geometrical Viewpoint, ICTP Trieste, World Scientific, 1991. MR **94c:**57040

[F] Benjamin Fine, *Algebraic theory of Bianchi groups*, Marcel-Dekker, 1989. MR **90h:**20002

[F-N] Benjamin Fine and Morris Newman, *The normal subgroup structure of the Picard group*, Trans. Amer. Math. Soc. **302** (1987), 769–786. MR **88d:**20070

[S] J.-P. Serre, *Le problème des groupes de congruence pour* SL_2, Ann. of Math. **92** (1970), 489–527. MR **42:**7671

[St] John Stallings, *Non-positively curved triangles of groups*, Group Theory from a Geometrical Viewpoint, ICTP trieste, World Scientific, 1991. MR **94b:**20033

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