PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 124, Number 10, October 1996

ON THE FAILURE OF CLOSE-TO-NORMAL STRUCTURE TYPE CONDITIONS AND PATHOLOGICAL KANNAN MAPS

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(Communicated by Palle E. T. Jorgensen)

ABSTRACT. We consider the failure of close-to-normal structure type conditions and show that a Banach space can be renormed to fail close-to-weak normal structure exactly when it contains a norm inseparable weakly compact subset. Included is an example of a particularly pathological fixed point free Kannan map.

1.

Throughout X will denote a real Banach space. We recall that X is said to have (weak) normal structure if whenever C is a closed (weak compact) bounded convex subset of X with $\operatorname{diam} C > 0$, then $\operatorname{rad} C < \operatorname{diam} C$ where

$$\operatorname{diam} C := \sup\{\|x-y\| : x, y \in C\} \quad \text{and} \quad \operatorname{rad} C := \inf_{x\in C} \sup\{\|x-y\| : y \in C\}$$

are the diameter and radius of the set C. We will denote normal structure and weak normal structure by, respectively, ns and w-ns. If X is a dual space, it has weak star normal structure (w*-ns) if we require the set C of the above definition to be weak star compact. A Banach space X has uniform normal structure if

$$\sup\left\{\frac{\operatorname{rad} C}{\operatorname{diam} C} : C \text{ nonempty nonsingleton closed bounded convex subset of } X\right\} < 1.$$

The above normal structure type conditions have been useful in the fixed point theory of nonexpansive maps (see [2] for example). A Banach space X has the fixed point property (weak fixed point property) if, given a nonempty closed (weak compact) bounded convex subset C of X that is self-mapped by a nonexpansive map T, then T has a fixed point in C (recall that $T : C \to C$ is nonexpansive if $\|Tx - Ty\| \le \|x - y\|$ for all $x, y \in C$). We abbreviate the two properties to fpp and w-fpp. If X is a dual space, then it has the weak star fixed point property (w*-fpp) if we require the set C of the above definition to be weak star compact. It is known that w-ns implies the w-fpp and that w*-ns implies the w*-fpp.

A Banach space X is said to have close-to-normal structure (close-to-ns) if, given a nonempty nonsingleton closed bounded convex subset C of X, there exists

Received by the editors February 23, 1995.
1991 *Mathematics Subject Classification.* Primary 46B20, 47H09.

$x \in C$ so that $\|x - y\| < \operatorname{diam} C$ for any $y \in C$. We can also define close-to-weak normal structure (close-to-w-ns) and (if X is a dual space) close-to-weak star normal structure (close-to-w*-ns) by replacing "closed bounded" in the above by "weak compact" or "weak star compact."

A self-map T of a subset C of a Banach space X is a Kannan map if

$$\|Tx - Ty\| \leq \frac{1}{2}(\|Tx - x\| + \|Ty - y\|)$$

for all $x, y \in C$. We will say that X has the weak fixed point property (w-fpp) for Kannan maps if, given a nonempty convex weak compact subset C of X that is self-mapped by a Kannan map T, then T has a fixed point. The w*-fpp for Kannan maps can be defined when X is a dual space by replacing "weak compact" with "weak star compact" in the above.

Wong [12] showed that X has the w-fpp for Kannan maps if and only if it has close-to-w-ns. The results stated in [6] give that close-to-w*-ns is equivalent to the w*-fpp for Kannan maps. It is shown in [11] that spaces which are separable or strictly convex have close-to-ns and that the KK property implies close-to-w-ns. This is seen by the use of a convex series. Suppose that C is a closed bounded nonempty and convex subset of a Banach space and that A is a countable subset of C. Then $\overline{\text{co}}A$ is a separable subset of C. Let $\{x_n\}_{n=1}^{\infty}$ be a countable dense subset of $\overline{\text{co}}A$. Consider the point $x = \Sigma x_n / 2^n$. Since C is a closed bounded convex subset of a Banach space, x exists and is in C. Now it is easily seen that if $y \in C$ (so $\|x_n - y\| \leq \operatorname{diam} C$ for all n) and $\|x - y\| = \operatorname{diam} C$, then $\|x_n - y\| = \operatorname{diam} C$ for all n. But this also gives that $\operatorname{dist}(y, \overline{\text{co}}A) = \operatorname{diam} C$. This procedure can be used to establish all of the results from [11] that were stated above.

In Section 2 we give examples of spaces that fail the close-to-normal structure conditions defined above, covering what appears to be known so far.

In Section 3 we concern ourselves with the equivalence stated in the abstract and give an example of a fixed point free idempotent Kannan map.

2.

Perhaps the simplest example of a set violating close-to-w-ns (and thus close-to-ns) is in $c_0(\Gamma)$ for an uncountable Γ. Indeed, if the x_i are the usual unit basis elements, then $C := \overline{\text{co}}\{x_i\}_{i \in \Gamma}$ is weak compact, convex, and, for any $x \in c_0(\Gamma)$ and $i \in \Gamma$, $\|x_i - x\| \geq 1$ if $i \notin \operatorname{supp} x$. Of course $\operatorname{diam} C = 1$.

It is not hard to give nonconstructive subsets of l_∞ violating close-to-ns, as in the following example.

Example 1. Let $\mathcal{U}$ be a free ultrafilter on $\mathbb{N}$. Put

$$A := \{x \in l_\infty : x(n) \in \{0, 1\} \text{ for } n \in \mathbb{N} \text{ and } \{n : x(n) = 0\} \in \mathcal{U}\}$$

and

$$C := \overline{\text{co}}A.$$

Clearly $\operatorname{diam} C = 1$. We show that for any $y \in C$ there exists $x \in A$ so that $\|y - x\| = 1$. Indeed, suppose that $y \in C$ and $m \in \mathbb{N}$. Define

$$B_m := \{n \in \mathbb{N} : y(n) \leq 1/m\}.$$

Note that if $z \in \operatorname{co} A$, $z = \lambda_1 x_1 + \lambda_2 x_2 + \cdots + \lambda_p x_p$, say, with $x_i \in A$, $\sum \lambda_i = 1, \lambda_i \geq 0$, then there exists $U \in \mathcal{U}$ so that $z(n) = 0$ for all $n \in U$. This will imply that the complement of B_m is not in $\mathcal{U}$, giving $B_m \in \mathcal{U}$.

It follows that there exists an infinite subsequence $(n_m)_{m\in\mathbb{N}}$ of $\mathbb{N}$ so that $y(n_m) \leq 1/m$ for all $m \in \mathbb{N}$. But every infinite subset of $\mathbb{N}$ contains an infinite subset that is not in $\mathcal{U}$. Thus, we can extract a further sequence $(n_{m_i})_{i\in\mathbb{N}}$ so that $y(n_{m_i}) \to_i 0$ and $M := \{n_{m_i}\}_{i\in\mathbb{N}} \notin \mathcal{U}$. Then $\chi_M \in A$ and $\|y - \chi_M\| = 1$, giving the result. $\square$

In [6] it is shown that l_∞ fails close-to-w*-ns. Indeed their result is that if (Ω, Σ, μ) is a sigma finite measure space (so that the dual of $L_1(\Omega, \Sigma, \mu)$ is $L_\infty(\Omega, \Sigma, \mu)$), then $L_\infty(\Omega, \Sigma, \mu)$ fails close-to-w*-ns if it is inseparable. Of course, l_∞ has close-to-w-ns by the result of Wong given earlier, since all of its weak compacts are (norm) separable. In [4] the above result from [6] is used to show that if X is an infinite dimensional Hilbert space, then $B(X)$, the space of bounded linear operators on X, fails close-to-w*-ns. It is also shown that the space of compact operators on X has close-to-w-ns if and only if X is inseparable but that the space of trace class operators always has close-to-w*-ns. It was subsequently shown in [5] that the trace class has w*-ns.

Suppose that Ω is a compact Hausdorff space. Then $C(\Omega)^*$, the space of continuous real valued functions on Ω with the supremum norm, fails close-to-w*-ns exactly when Ω is uncountable (that is, when $C(\Omega)^*$ is inseparable). To verify this, we first identify $C(\Omega)^*$ with $M(\Omega)$, the space of radon measures on Ω with total variation norm, the actions on $C(\Omega)$ being integration. If Ω is countable, to show that $C(\Omega)^*$ has close-to-w*-ns we can assume that Ω is infinite. Then $M(\Omega) \equiv l_1$, a separable space which thus has close-to-w*-ns from above. If Ω is uncountable, consider

$$C := \{\mu \in M(\Omega) : \|\mu\| \leq 1,\ \mu(\Omega) = 1\}.$$

C is the intersection of the w* compact unit ball and a w* closed hyperplane and so is thus convex and w* compact. Clearly $\operatorname{diam} C = 2$. If $\mu \in C$, then $\{x \in \Omega : \mu(\{x\}) = 0\} \neq \varnothing$ since Ω is uncountable. Now if δ_x denotes the dirac measure at x, then $\|\mu - \delta_x\| = 2$. Thus $C(\Omega)^*$ fails close-to-w*-ns.

We make a slight digression here on the failure of w*-ns and the w*-fpp. Suppose that K is a locally compact Hausdorff space with $C_0(K)$ denoting the space of real valued continuous functions on K vanishing at infinity. In [4] it was asked when does $C_0(K)^*$ fail w*-ns? We note here that $C_0(K)^*$ fails w*-ns (and also the w*-fpp) exactly when K is nondiscrete. To verify this, we identify $C_0(K)^*$ with $M(K)$, the space of radon measures on K. As in [9], if Ω is a compact subset of K, then $C(\Omega)^*$ is isometrically isomorphic to a w* closed subspace of $C_0(K)^*$ via a w* homeomorphism (namely, the map which extends a measure to be identically zero outside Ω). Now if K is nondiscrete, then, by the local compactness, it contains an infinite compact subset Ω. But $C(\Omega)^*$ fails the w*-fpp by a result from [9], and thus so does $C_0(K)^*$. Otherwise, if K has the discrete topology, then $C_0(K)^* \equiv c_0(K)^*$, well known to have w*-ns (and thus the w*-fpp).

We now give a general method for producing spaces which fail close-to-ns before considering renorming results. Suppose that the Banach space X fails uniform normal structure. That is, for every $m \in \mathbb{N}$ there exists a closed convex subset C_m of X so that $\operatorname{diam} C_m = 1$ and $\operatorname{rad} C_m \geq 1 - 1/m$. We can also assume that $0 \in C_m$ for all m. Put $Y := l_\infty(X)$ and

$$C := \prod_m C_m = \{(x_m) \in l_\infty(X) : x_m \in C_m \text{ for all } m\}.$$

Clearly C is closed, convex and $\operatorname{diam} C = 1$. Now suppose that $(x_m) \in C$. For any m there exists $y_m \in C_m$ so that $\|x_m - y_m\| > 1 - 2/m$. Then $\|(x_m) - (y_m)\| = 1$, showing that Y fails close-to-ns.

The above example can be further refined using ultrapowers. For material on ultrapowers see, for example, [3], [7] or [2]. Suppose X is as above and $\mathcal{U}$ is a free ultrafilter on $\mathbb{N}$. Let $\widetilde{X}$ be the associated ultrapower of X, so $\widetilde{X} = l_\infty(X)/\mathcal{N}(\mathcal{U})$, where

$$\mathcal{N}(\mathcal{U}) := \{(x_n) \in l_\infty(X) : \lim_{\mathcal{U}} \|x_n\| = 0\}.$$

Now define

$$\widetilde{C} := \left(\prod_m C_m\right)_{\mathcal{U}} = \{(x_m)_{\mathcal{U}} \in \widetilde{X} : x_m \in C_m \text{ for all } m\}.$$

Then $\widetilde{C}$ is closed, convex and bounded with $\operatorname{diam} \widetilde{C} = 1$. Also if $(x_m)_{\mathcal{U}} \in \widetilde{C}$ and we choose the y_m as above, then $\|x_m - y_m\| \to 1$, so $\|(x_m)_{\mathcal{U}} - (y_m)_{\mathcal{U}}\|_{\mathcal{U}} = \lim_{\mathcal{U}} \|x_m - y_m\| = 1$, showing that $\widetilde{X}$ fails close-to-ns.

Of course if X was originally superreflexive (so every ultrapower of X is also superreflexive), then $\widetilde{X}$ is a superreflexive space that fails close-to-ns. We note that superreflexive spaces that fail uniform normal structure are easily given using a result from [1]: Every infinite-dimensional Banach space can be renormed to fail normal structure. More examples of superreflexive spaces failing close-to-ns are given below, where we are concerned with renorming spaces to fail close-to-ns or close-to-w-ns. The pathological sets so obtained will be similar to the example in $c_0(\Gamma)$ given above.

We start by recalling that a class of pairs $(x_i, x_i^*)_{i \in I}$, where $x_i \in X$ and $x_i^* \in X^*$, is called a biorthogonal system if $x_i^*(x_j) = \delta_i^j$ for any $i, j \in I$. The following proposition uses an adaptation of the technique for renorming to fail normal structure used in [1].

Proposition 2. *Suppose that X admits an uncountable biorthogonal system. Then X can be renormed to fail close-to-ns.*

Proof. Suppose $(x_i, x_i^*)_{i \in I}$ is the uncountable system. We can assume that the set $\{x_i\}_{i \in I}$ is bounded. Now, since I is uncountable, there exists $N \in \mathbb{N}$ and an uncountable subset J of I so that $\|x_i^*\| \leq N$ for all $i \in J$. For simplicity we will assume that $I = J$. Now for $x \in X$ define

$$\|x\|' := \max\left\{\frac{\|x\|}{\operatorname{diam}\{x_i\}_{i \in I}},\ \sup_{i \in I} |x_i^*(x)|\right\}.$$

Obviously $(X, \| \|')$ is isomorphic to X. Put $C := \overline{\text{co}}\{x_i\}_{i \in I}$. Clearly $\operatorname{diam} C = 1$. Now suppose that $x \in \overline{\text{span}}\{x_i\}_{i \in I}$. Then there exists a countable subset A of I so that $x \in \overline{\text{span}}\{x_i\}_{i \in A}$. Thus, if we choose $j \in I \backslash A$, then $\|x_j - x\|' \geq |x_j^*(x_j - x)| = 1$. This obviously implies that C violates close-to-ns. $\square$

If $X = l_2(I)$, I uncountable, with the x_i the usual unit basis elements and x_i^* the coordinate functionals, then, in the above,

$$\|x\|' = \max\left\{\frac{\|x\|_2}{2^{1/2}}, \|x\|_\infty\right\},$$

giving $Y := (X, \| \|')$ isometric to an example used in [10], the norms differing by a multiplication by $2^{1/2}$. As in [10], the set $\{x \in Y : x(i) \geq 0 \text{ for all } i \text{ and } \|x\|_2 \leq 1\}$ can be used instead of $\overline{\text{co}}\{x_i\}$ to show failure of close-to-ns.

Also, note that if $X = c_0(I), (x_i, x_i^*)_{i \in I}$ the usual biorthogonal system, then $\| \|' = \| \|_\infty$.

Not every inseparable Banach space has an uncountable biorthogonal system (see pg. 861 of [8]). It is unknown whether every inseparable Banach space can be renormed to fail close-to-ns. For the analogous close-to-w-ns problem the answer is given in the next section.

3.

First we recall some further material. A set $\{x_i\}_{i \in I}$ is said to be an M-basis of a Banach space X if $\overline{\text{span}}\{x_i\}_{i \in I} = X$ and there exists a total family $\{x_i^*\}_{i \in I} \subseteq X^*$ so that $(x_i, x_i^*)_{i \in I}$ is a biorthogonal system. Let $\delta(X)$ denote the density character of X. That is,

$$\delta(X) = \min\{k : X \text{ has a dense subset of cardinality } \leq k\}.$$

In general, if $(x_i, x_i^*)_{i \in I}$ is a biorthogonal system, then $|I| \leq \delta(X)$ (see pg. 673 of [8]), and if $\{x_i\}_{i \in I}$ is an M-basis of X, then clearly $|I| = \delta(X)$ (assuming X is infinite dimensional).

Theorem 3. *A Banach space X can be renormed to fail close-to-w-ns if and only if it contains a (norm) inseparable weak compact subset.*

Proof. The forward implication is clear since any (norm) isomorphism is a weak homeomorphism.

For the reverse implication suppose that X contains an inseparable weak compact set C. Then $Y := \overline{\text{span}} C$ is an inseparable weakly compactly generated subspace of X. By a well-known result, Y can then be generated by a weak compact balanced convex set K.

Theorem 20.5(a) on pg. 693 of [8] will now give that Y has an M-basis $\{x_i\}_{i \in I}$ with $x_i \in K$ for all i.

Since $\delta(Y) = |I|$, I is uncountable. Since K is weak compact, $\{x_i\}_{i \in I}$ is bounded. We can also extend the functionals associated with the M-basis to all of X by Hahn-Banach and use the method of proof of the above proposition to produce an uncountable subset J of I and a renorming of X so that the weak compact set $\overline{\text{co}}\{x_i\}_{i \in J}$ violates close-to-w-ns. ☐

The following corollary is now immediate.

Corollary 4. *If X is an inseparable weakly compactly generated space, then it can be renormed to fail close-to-w-ns.*

A self-map T of a nonempty set C is periodic if there exists $n \in \mathbb{N}$ so that $T^n = I$. If C is a nonempty subset of a Banach space X and $T : C \to C$ is nonexpansive, then T is said to be rotative if there exist $n \in \mathbb{N}$ and $a < n$, $a \in \mathbb{R}$, so that

$$\|x - T^n x\| \leq a\|x - Tx\| \text{ for all } x \in C.$$

Clearly a periodic nonexpansive map is rotative. Using the fixed point result concerning rotative maps that is given on pg. 177 of [2], we have that if C is a nonempty closed convex subset of X, then any periodic nonexpansive self-map of C has a fixed point. However, this is not true for periodic Kannan mappings.

Proposition 5. *Suppose that X admits a biorthogonal system $(x_i, x_i^*)_{i\in I}$ with $|I| = 2^{\aleph_0}$. Then X can be renormed so that the resulting space contains a nonempty bounded closed convex subset C on which a fixed point free self-mapping Kannan map T is defined satisfying $T^2 = I$.*

Proof. Since the cofinality of $2^{\aleph_0}$ is uncountable, there exists a subset J of I so that $|J| = 2^{\aleph_0}$ and $N \in \mathbb{N}$ so that $\|x_i\| \le N$ and $\|x_i^*\| \le N$ for all $i \in J$.

We renorm X as in the proof of Proposition 2, with

$$\|x\|' = \max\left\{\frac{\|x\|}{\operatorname{diam}\{x_i\}_{i\in J}},\ \sup_{i\in J}|x_i^*(x)|\right\}.$$

With $C := \overline{\mathrm{co}}\{x_i\}_{i\in J}$ we note that the proof of Proposition 2 gives that if $x \in C$, then $|\{i \in J : \|x_i - x\| = 1\}| = 2^{\aleph_0}$. Also, since $(2^{\aleph_0})^{\aleph_0} = 2^{\aleph_0}$, it follows that $|C| = 2^{\aleph_0}$. Write $C = \{y_i\}_{i<2^{\aleph_0}}$. Define $T : C \to C$ recursively as follows. Suppose that T has been defined on $\{y_i\}_{i<\alpha}$ for some ordinal $\alpha < 2^{\aleph_0}$.

Case 1. There exists $j < \alpha$ so that $Ty_j = y_\alpha$. Then define $Ty_\alpha := y_j$.

Case 2. Otherwise. Put $F := \{y_i\}_{i<\alpha} \cup \{Ty_i\}_{i<\alpha}$. Then $|F| < 2^{\aleph_0}$. From above, there exists $z \in C \setminus F$ so that $\|y_\alpha - z\| = 1$. Now define $Ty_\alpha := z$.

Note that in the above procedure the j in Case 1 is unique. Also $\|Tx - x\| = 1$ for all $x \in C$ and $\operatorname{diam} C = 1$, so that T is a fixed point free Kannan self-map of C. Finally, $T^2 = I$. Indeed, suppose that $\alpha < 2^{\aleph_0}$. First, suppose $Ty_\alpha = y_j$ for some $j < \alpha$. Then $Ty_j = y_\alpha$ from the definition of T, so $T^2 y_\alpha = y_\alpha$. Otherwise, $Ty_\alpha = y_j$ for some $j > \alpha$. Then, again from the definition of T, $T(y_j) = y_\alpha$ also giving $T^2 y_\alpha = y_\alpha$. □

By combining the methods of proof of Theorem 3 and Proposition 5, we obtain the following.

Corollary 6. *Suppose X contains a weak compact subset K with $\delta(K) \ge 2^{\aleph_0}$. Then X can be renormed so that the resulting space contains a nonempty weak compact convex subset D on which a self-mapping fixed point free Kannan map is defined satisfying $T^2 = I$.*

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