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ISOMETRIES OF CERTAIN OPERATOR ALGEBRAS

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ABSTRACT. To a given basis $\phi_1, \dots, \phi_n$ on an n-dimensional Hilbert space $\mathcal{H}$, we associate the algebra $\mathfrak{A}$ of all linear operators on $\mathcal{H}$ having every ϕ_j as an eigenvector. So, $\mathfrak{A}$ is commutative, semisimple, and n-dimensional. Given two algebras of this type, $\mathfrak{A}$ and $\mathfrak{B}$, there is a natural algebraic isomorphism τ of $\mathfrak{A}$ and $\mathfrak{B}$. We study the question: When does τ preserve the operator norm?

0. INTRODUCTION

We consider an n-dimensional Hilbert space $\mathcal{H}$ and fix an ordered basis $\phi_1, \dots, \phi_n$ in $\mathcal{H}$. For each $\lambda = (\lambda_1, \dots, \lambda_n)$ in $\mathbf{C}^n$, we denote by T_λ the operator in $\mathcal{B}(\mathcal{H})$ with $T_\lambda \phi_j = \lambda_j \phi_j$, $1 \le j \le n$. We put $\mathfrak{A} = \{T_\lambda | \lambda \in \mathbf{C}^n\}$. Thus $\mathfrak{A}$ is an n-dimensional commutative semisimple operator algebra on $\mathcal{H}$.

Given two such algebras $\mathfrak{A}$ and $\mathfrak{B}$ on Hilbert spaces $\mathcal{H}$ and $\mathcal{H}'$, there is a natural map τ of $\mathfrak{A}$ on $\mathfrak{B}$, described as follows:

Let us denote by P_j the idempotent operator in $\mathfrak{A}$ such that $P_j \phi_j = \phi_j$, $P_j \phi_k = 0$ if $k \ne j$, $1 \le j$, $k \le n$. Then, for each λ,

$$T_\lambda = \sum_{j=1}^n \lambda_j P_j.$$

Given two such algebras $\mathfrak{A}$ and $\mathfrak{B}$ such that $\mathfrak{A}$ corresponds to the idempotents P_j and $\mathfrak{B}$ corresponds to the idempotents Q_j, we define the map τ of $\mathfrak{A}$ on $\mathfrak{B}$ by

$$\tau\left(\sum_{j=1}^n \lambda_j P_j\right) = \sum_{j=1}^n \lambda_j Q_j.$$

Then, τ is an algebraic isomorphism of $\mathfrak{A}$ on $\mathfrak{B}$ and $\tau(P_j) = Q_j$, for each j. We are interested in the following questions:

(a) Under what conditions is τ an isometric map from $\mathfrak{A}$ to $\mathfrak{B}$ taken in the operator norm?
(b) Assuming that τ is isometric, is τ induced by a unitary equivalence of $\mathfrak{A}$ and $\mathfrak{B}$?

We shall prove the following results:

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Theorem 1. *Let $\mathcal{H}$ be an n-dimensional Hilbert space. Fix $S, T \in \mathcal{B}(\mathcal{H})$ with* rank $T = 1$. *Then*

$$\|T + zS\| = \|T\| + \|T\|^{-1} \operatorname{Re}[z \text{ trace}(T^*S)] + o(z), \qquad z \in \mathbf{C}. \tag{1}$$

Consider next algebras $\mathfrak{A}$ and $\mathfrak{B}$ corresponding to idempotents P_j and Q_j, and let τ be the isomorphism of $\mathfrak{A}$ on $\mathfrak{B}$. The following theorem gives an answer to question (a) above.

Theorem 2. *If τ is isometric, then, writing* tr *for* trace*,*

$$\operatorname{tr}(P_i^* P_j) = \operatorname{tr}(Q_i^* Q_j) \quad \textit{for all } i, j. \tag{2}$$

If $n = 3$, then (2) *is sufficient for τ to be isometric.*

We shall use Theorem 1 in the proof of Theorem 2. The next theorem is due to Warren Wogen.

Theorem 3. *There exist algebras $\mathfrak{A}$ and $\mathfrak{B}$ such that the isomorphism τ is isometric, but τ is not induced by a unitary map on the underlying Hilbert spaces.*

The final result concerns an interesting class of algebras for which the answer is positive. These algebras arise in the work of Sarason [3] on Pick interpolation, and are isometrically isomorphic as Banach algebras to quotients of the disk algebra by a closed ideal. They can be described as operator algebras in several ways, and we do it as follows: As in [1], we fix an n-tuple $\alpha = (\alpha_1, \dots, \alpha_n)$ of distinct points in the open unit disk and associate to α an inner product $(\ ,\ )_\alpha$ on $\mathbf{C}^n$ by putting

$$(t, s)_\alpha = \sum_{j,k=1}^{n} t_j \overline{s}_k (1 - \alpha_j \overline{\alpha}_k)^{-1}, \qquad t, s \in \mathbf{C}^n.$$

We denote the resulting inner product space by $\mathbf{C}^n_\alpha$. For each $w = (w_1, \dots, w_n)$ in $\mathbf{C}^n$ we define the operator R_w on $\mathbf{C}^n_\alpha$ by

$$R_w(t) = (t_1 w_1, \dots, t_n w_n), \qquad t = (t_1, \dots, t_n) \in \mathbf{C}^n.$$

We denote the algebra consisting of all operators R_w with $w \in \mathbf{C}^n$ by $\mathfrak{A}_\alpha$ and we call $\mathfrak{A}_\alpha$ the *Pick algebra corresponding to α*. The standard basis of $\mathbf{C}^n$ then consists of simultaneous eigenvectors of the operators in $\mathfrak{A}_\alpha$, and so $\mathfrak{A}_\alpha$ belongs to the class of algebras introduced earlier. We studied Pick algebras in [1] and shall use some results obtained there. We shall show:

Theorem 4. *Fix α as above and form the algebra $\mathfrak{A}_\alpha$. If $\mathfrak{B}$ is another algebra of the type considered such that the map $\tau : \mathfrak{A}_\alpha \mapsto \mathfrak{B}$ is isometric, then τ is induced by a unitary map of $\mathbf{C}^n_\alpha$ on the Hilbert space $\mathcal{H}$ where $\mathfrak{B}$ is defined.*

1. Proofs

Proof of Theorem 1. We denote by M_n the space of all $n \times n$ matrices with complex entries. For $T, S \in M_n$, we put

$$(T, S)_0 = \operatorname{tr}(TS^*).$$

The corresponding norm $\|T\|_0$ satisfies

$$\|T\|_0^2 = \operatorname{tr}(TT^*).$$

We write $\|T\|$ for the operator norm of T on $\mathbf{C}^n$, taken in the Euclidean norm.

Claim 1. Fix $T, S \in M_n$ with $\|T\|_0 = 1$. Then, for $z \in \mathbf{C}$,

$$\|T + zS\|_0 = 1 + \mathrm{Re}[z \ \mathrm{tr}(ST^*)] + O(z^2).$$

Proof of Claim 1. The formula follows from the fact that $\| \ \|_0$ is induced by an inner product on M_n.

Claim 2. Fix $T \in M_n$. Then $\|T\| \le \|T\|_0$.

Proof of Claim 2. Let $\lambda_1, \dots, \lambda_n$ be the eigenvalues of TT^*, $0 \le \lambda_1 \le \cdots \le \lambda_n$. Then

$$\|TT^*\| = \lambda_n \le \sum_{j=1}^{n} \lambda_j = \mathrm{tr}(TT^*),$$

and so $\|T\|^2 \le \|T\|_0^2$.

Claim 3. Fix $T \in M_n$ with rank $T = 1$. Then $\|T\| = \|T\|_0$.

Proof of Claim 3. WLOG $\|T\| = 1$. So, $\|TT^*\| = \|T\|^2 = 1$. Also, $TT^* \ge 0$. Since rank $T = 1$, $\mathrm{rank}(TT^*) = 1$. It follows that $\mathrm{tr}(TT^*) = 1$. So, $\|T\|_0 = \|T\|$, as claimed.

We next require the following elementary lemma.

Lemma 1. *Let f, g be convex real-valued functions defined on $\mathbf{R}^N$ such that*

(i) *$f(x) \le g(x)$ for all x,*
(ii) *$f(0) = g(0)$, and*
(iii) *g is differentiable at* 0.

Then, f is differentiable at 0 and f and g have the same linear part at 0.

To prove Theorem 1, we now fix operators S and T in $\mathcal{B}(\mathcal{H})$ such that rank $T = 1$ and $\|T\| = 1$. We define the functions

$$\phi_0(z) = \|T + zS\|_0 \quad \text{and} \quad \phi(z) = \|T + zS\|, \qquad z \in \mathbf{C}.$$

Thus ϕ_0 and ϕ are convex functions defined in the z-plane. By Claims 2 and 3 we have

$$\phi(z) \le \phi_0(z), \qquad z \in \mathbf{C}, \tag{3}$$

and

$$\phi(0) = \phi_0(0) = 1. \tag{4}$$

Applying the lemma to the functions ϕ and ϕ_0 which are defined on $\mathbf{R}^2$, and using the fact that by Claim 1

$$\phi_0(z) = 1 + \mathrm{Re}[z \ \mathrm{tr}(ST^*)] + O(z^2)$$

we conclude that

$$\phi(z) = 1 + \mathrm{Re}[z \ \mathrm{tr}(ST^*)] + o(z),$$

and so

$$\|T + zS\| = 1 + \mathrm{Re}[z \ \mathrm{tr}(ST^*)] + o(z). \tag{5}$$

If we drop the assumption that $\|T\| = 1$, we may apply (5) to $\|T\|^{-1}T$ and $\|T\|^{-1}S$, and obtain

$$\|T + zS\| = \|T\| + \|T\|^{-1}\mathrm{Re}[z \ \mathrm{tr}(ST^*)] + o(z). \tag{6}$$

Theorem 1 is proved. □

Proof of Theorem 2. We are given an algebra $\mathfrak{A}$ spanned by the idempotents P_j, $1 \le j \le n$, and an algebra $\mathfrak{B}$ spanned by the idempotents Q_j, $1 \le j \le n$. We assume that the isomorphism τ of $\mathfrak{A}$ on $\mathfrak{B}$, which takes P_j to Q_j for each j, is isometric. For given i, j, we have by Theorem 1 that

$$\|P_i + zP_j\| = \|P_i\| + \|P_i\|^{-1}\mathrm{Re}[z \ \mathrm{tr}(P_i^* P_j)] + o(z)$$

and

$$\|Q_i + zQ_j\| = \|Q_i\| + \|Q_i\|^{-1}\mathrm{Re}[z \ \mathrm{tr}(Q_i^* Q_j)] + o(z).$$

Since τ is isometric, the left-hand sides are equal, and hence the right-hand sides are equal, and so the linear terms on the right-hand sides coincide. It follows that

$$\mathrm{tr}(P_i^* P_j) = \mathrm{tr}(Q_i^* Q_j) \quad \text{for all } i, j,$$

i.e., (2) holds. Thus (2) is a necessary condition for τ to be isometric.

We now assume that $n = 3$. We choose algebras $\mathfrak{A}$ and $\mathfrak{B}$ so that (2) holds. For each $\lambda \in \mathbf{C}^3$, we put

$$S_\lambda = \sum_{j=1}^{3} \lambda_j P_j \in \mathfrak{A} \quad \text{and} \quad T_\lambda = \sum_{j=1}^{3} \lambda_j Q_j \in \mathfrak{B}.$$

Then, $\tau(S_\lambda) = T_\lambda$. Fix now $\lambda = (\lambda_1, \lambda_2, \lambda_3)$ with $\lambda_j \ne 0$ for each j.

We denote by μ_1, μ_2, μ_3 the eigenvalues of $S_\lambda^* S_\lambda$ and by μ_1', μ_2', μ_3' the eigenvalues of $T_\lambda^* T_\lambda$. By (2),

$$\mathrm{tr}(S_\lambda^* S_\lambda) = \sum_{i,j} \overline{\lambda}_i \lambda_j \ \mathrm{tr}(P_i^* P_j) = \sum_{i,j} \overline{\lambda}_i \lambda_j \ \mathrm{tr}(Q_i^* Q_j) = \mathrm{tr}(T_\lambda^* T_\lambda).$$

Thus

$$\sum_i \mu_i = \sum_i \mu_i'. \tag{7}$$

We use coordinatewise multiplication in $\mathbf{C}^3$, so $1/\lambda = (1/\lambda_1, 1/\lambda_2, 1/\lambda_3)$. Then

$$(S_\lambda^* S_\lambda)^{-1} = S_\lambda^{-1}(S_\lambda^{-1})^* = S_{1/\lambda}(S_{1/\lambda})^*,$$
$$\mathrm{tr}[(S_\lambda^* S_\lambda)^{-1}] = \mathrm{tr}[S_{1/\lambda}(S_{1/\lambda})^*] = \mathrm{tr}[T_{1/\lambda}(T_{1/\lambda})^*] = \mathrm{tr}[(T_\lambda^* T_\lambda)^{-1}].$$

The eigenvalues of $(S_\lambda^* S_\lambda)^{-1}$ are $1/\mu_1, 1/\mu_2, 1/\mu_3$ and similarly for $(T_\lambda^* T_\lambda)^{-1}$. So

$$\sum_i 1/\mu_i = \sum_i 1/\mu_i'. \tag{8}$$

Next we note that for all λ

$$\det(S_\lambda) = \lambda_1 \lambda_2 \lambda_3 = \det(T_\lambda),$$

and so we have

$$\det(S_\lambda^* S_\lambda) = |\det S_\lambda|^2 = |\det T_\lambda|^2 = \det(T_\lambda^* T_\lambda),$$

so

$$\mu_1 \mu_2 \mu_3 = \mu_1' \mu_2' \mu_3'. \tag{9}$$

Hence (8) is equivalent to

$$\mu_2 \mu_3 + \mu_1 \mu_3 + \mu_1 \mu_2 = \mu_2' \mu_3' + \mu_1' \mu_3' + \mu_1' \mu_2'. \tag{10}$$

Equations (7), (9), and (10) give us that the triples μ_1, μ_2, μ_3 and μ'_1, μ'_2, μ'_3 have the same elementary symmetric functions, and so it follows that

$$\max \mu_j = \max \mu'_j.$$

Thus $\|S^*_\lambda S_\lambda\| = \|T^*_\lambda T_\lambda\|$, and hence $\|S_\lambda\| = \|T_\lambda\|$. This holds for all λ with $\lambda_j \neq 0$ for all j. Hence, τ is isometric. Theorem 2 is proved. □

Proof of Theorem 3 (Wogen's example). Choose an orthonormal basis E_1, E_2, E_3 for a 3-dimensional Hilbert space $\mathcal{H}$. Define vectors ϕ_1, ϕ_2, ϕ_3 by

$$\phi_1 = E_1, \quad \phi_2 = -E_1 + E_2, \quad \phi_3 = -E_2 + E_3.$$

The ϕ_j form a basis for $\mathcal{H}$. The dual basis ψ_1, ψ_2, ψ_3, which satisfies $(\phi_j, \psi_k) = \delta^k_j$, is given by

$$\psi_1 = E_1 + E_2 + E_3, \quad \psi_2 = E_2 + E_3, \quad \psi_3 = E_3.$$

We define the idempotent operators P_j, $1 \leq j \leq 3$, by

$$P_j \phi_k = \delta^k_j \phi_k, \qquad 1 \leq k \leq 3,$$

and we similarly define Q_j, $1 \leq j \leq 3$, by

$$Q_j \psi_k = \delta^k_j \psi_k, \qquad 1 \leq k \leq 3.$$

We denote by $\mathfrak{A}$ the algebra spanned by the P_j and by $\mathfrak{B}$ the algebra spanned by the Q_j. Let τ be the corresponding map: $\mathfrak{A} \mapsto \mathfrak{B}$.

Since the ψ_j form a dual basis for the ϕ_j, we have that

$$Q^*_j = P_j, \qquad 1 \leq j \leq 3.$$

Hence for all j, k

$$\mathrm{tr}(Q_j Q^*_k) = \mathrm{tr}(P^*_j P_k) = \mathrm{tr}(P_k P^*_j).$$

Also each P_j has a matrix, relative to the basis E_1, E_2, E_3, with real entries. Hence,

$$\mathrm{tr}(P_k P^*_j) = \mathrm{tr}[(P_k P^*_j)^*] = \mathrm{tr}(P_j P^*_k).$$

So,

$$\mathrm{tr}(Q_j Q^*_k) = \mathrm{tr}(P_j P^*_k).$$

Theorem 2 now gives that τ is isometric.

On the other hand, suppose that there exists a unitary transformation U of $\mathcal{H}$ on itself which induces τ. Then

$$U P_j U^{-1} = Q_j, \qquad j = 1, 2, 3.$$

Fix j. Then

$$U P_j U^{-1}(\psi_j) = Q_j \psi_j = \psi_j, \quad \text{so } P_j(U^{-1}\psi_j) = U^{-1}\psi_j.$$

It follows that there exist constants c_j such that

$$U^{-1}\psi_j = c_j \phi_j.$$

Then

$$(\psi_j, \psi_k) = (U^{-1}\psi_j, U^{-1}\psi_k) = c_j \overline{c}_k (\phi_j, \phi_k) \quad \text{for all } j, k.$$

But, by construction,

$$(\phi_1, \phi_3) = 0 \quad \text{and} \quad (\psi_1, \psi_3) = 1.$$

This is a contradiction and it follows that U cannot exist. We have shown: For the algebras $\mathfrak{A}$ and $\mathfrak{B}$, as given, τ is isometric and is not induced by a unitary map. This proves Theorem 3. $\square$

Proof of Theorem 4. The following lemma is implicitly proved in [1]. For convenience, we give a direct, explicit proof here.

Lemma 2. *Let $\mathcal{H}$ be an n-dimensional Hilbert space and let B be a linear transformation on $\mathcal{H}$ with eigenvectors $\nu_1, \ldots, \nu_n$ and corresponding eigenvalues $z_1, \ldots, z_n$. Assume that*

$$(\nu_j, \nu_k) = \frac{1}{1 - z_j \overline{z}_k} \quad \text{for } 1 \leq j, k \leq n.$$

Then $\|B^p\| = 1$, $p = 1, \ldots, n-1$.

Proof. Fix $t = \sum_{j=1}^n t_j \nu_j$ in $\mathcal{H}$. For $p = 1, 2, \ldots,$

$$\begin{aligned}((B^p)^* B^p t, t) = (B^p t, B^p t) &= \left(\sum_{j=1}^n z_j^p t_j \nu_j, \sum_{k=1}^n z_k^p t_k \nu_k \right) \\ &= \sum_{j,k} z_j^p \overline{z}_k^p t_j \overline{t}_k (\nu_j, \nu_k).\end{aligned}$$

Thus

$$\begin{aligned}((I - (B^p)^* B^p) t, t) &= \sum_{j,k} \left(\frac{1 - z_j^p \overline{z}_k^p}{1 - z_j \overline{z}_k} \right) t_j \overline{t}_k \\ &= \sum_{j,k} (1 + z_j \overline{z}_k + \cdots + z_j^{p-1} \overline{z}_k^{p-1}) t_j \overline{t}_k \\ &= \left| \sum_{j=1}^n t_j \right|^2 + \left| \sum_{j=1}^n z_j t_j \right|^2 + \cdots + \left| \sum_{j=1}^n z_j^{p-1} t_j \right|^2 .\end{aligned}$$

If $p \leq n-1$, we can choose $t = (t_1, \ldots, t_n) \neq 0$ such that each term vanishes and so

$$\|t\|^2 = \|B^p t\|^2,$$

and hence $\|B^p\| \geq 1$. Also, for every $t \in \mathbf{C}^n$, setting $p = 1$ gives

$$((I - B^* B) t, t) = \left| \sum_{j=1}^n t_j \right|^2 \geq 0,$$

and so $\|t\|^2 \geq \|Bt\|^2$. Hence $\|B\| \leq 1$, and so $\|B^p\| \leq 1$ for all p. It follows that

$$\|B^p\| = 1 \quad \text{for } p = 1, \ldots, n-1,$$

as claimed. $\square$

We note that, in the notation used in the paragraph following the statement of Theorem 3, we can apply Lemma 2 to the following situation: The Hilbert space $\mathcal{H} = \mathbf{C}^n_\alpha$ where α is a fixed n-tuple in $\mathbf{C}^n$, the operator B is the operator R_α, the eigenvectors ν_j are the standard basis in $\mathbf{C}^n$ and the corresponding eigenvalues are given by $z_j = \alpha_j$, $1 \le j \le n$. Then, by definition of the inner product $(\ , \ )_\alpha$, we have that

$$(\nu_j, \nu_k) = \frac{1}{1 - z_j \overline{z}_k}$$

as required in Lemma 2. Hence we conclude that

$$\|R_\alpha^p\| = 1, \qquad p = 1, \ldots, n-1.$$

In [1], we proved a converse to Lemma 2 (Lemma 4 in [1]), and related results are given in Lotto [2].

Converse. *Let $\mathcal{H}$ be an n-dimensional Hilbert space, and let B be an operator on $\mathcal{H}$ with eigenvalues $z_1, \ldots, z_n$ such that $z_j \ne z_k$ if $j \ne k$, and $|z_j| < 1$ for all j. Assume that $\|B^p\| = 1$, $p = 1, \ldots, n-1$. Then there exists a unitary map U of $\mathcal{H}$ on $\mathbf{C}_z^n$ such that*

$$R_z = UBU^{-1}.$$

For each $w \in \mathbf{C}^n$, R_w is the operator in $\mathfrak{A}_\alpha$ with eigenvalues $w_1, \ldots, w_n$. Let T_w denote the operator in the algebra $\mathfrak{B}$ with the same eigenvalue set w. Then the isomorphism τ of $\mathfrak{B}$ on $\mathfrak{A}_\alpha$ maps T_w on R_w. Since τ is isometric,

$$\|R_w\| = \|T_w\|$$

for all w. In particular

$$\|T_\alpha^p\| = \|R_\alpha^p\| = 1, \qquad p = 1, \ldots, n-1.$$

By the Converse above, there exists a unitary operator U from $\mathcal{H}$ to $\mathbf{C}_\alpha^n$ with

$$R_\alpha = UT_\alpha U^{-1}.$$

Fix $w^0 \in \mathbf{C}^n$. We choose a polynomial g such that $g(\alpha) = w^0$, in the sense that $g(\alpha_j) = w_j^0$ for each j. Then $g(T_\alpha) = T_{w^0}$ and

$$\begin{aligned}\tau(T_{w^0}) &= \tau[g(T_\alpha)] = g(\tau(T_\alpha)) = g(R_\alpha) \\ &= g(UT_\alpha U^{-1}) = Ug(T_\alpha)U^{-1} = UT_{w^0}U^{-1}.\end{aligned}$$

This holds for all w^0. Hence τ is induced by the unitary map U. Theorem 4 is proved. □

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