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INFINITE CYCLIC VERBAL SUBGROUPS OF RELATIVELY FREE GROUPS

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Abstract. We prove that there exist a relatively free group H and a word $w(x,y)$ in two variables such that the verbal subgroup of H defined by $w(x,y)$ is an infinite cyclic group whereas $w(x,y)$ has only one nontrivial value in H.

In [6] (see also [2], Problem 2.45a), the following question of P. Hall is stated. Is it true that if there is only finitely many different values of a word v in a group H, then the verbal subgroup $v(H)$ is finite? S. Ivanov [1] answered this question in the negative; he constructed a group H and a word $v(x,y)$ such that $v(H)$ is infinite cyclic but $v(x,y)$ has only one non-trivial value in H. In [1], he also asked about what may happen if extra conditions are imposed on H. In particular, he raised the problem as to whether the fact that there are finitely many values of a word v in a relatively free group H implies the finiteness of the verbal subgroup $v(H)$. He repeated this question later in [3]. It is interesting to note that the construction described in [5] shows that there exists a word $u(x,y)$ and a relatively free group K such that $u(K)$ is infinite cyclic and all the elements of $u(K)$ are values of the word $u(x,y)$ in K. In this paper we give the negative answer to the question raised by S. Ivanov.

We put

$$u = u(x,y) = (x^d y^d)^d x^d,$$
$$v = v(x,y) = [u^d, x^d],$$
$$w(x,y) = u^{\varepsilon_1} v^{n+1} u^{\varepsilon_2} v^{n+2} \cdots u^{\varepsilon_h} v^{n+h},$$

where $h \equiv 0 \,(\mathrm{mod}\ 10)$, $\varepsilon_{10k+1} = \varepsilon_{10k+2} = \varepsilon_{10k+3} = \varepsilon_{10k+5} = \varepsilon_{10k+6} = 1$, $\varepsilon_{10k+4} = \varepsilon_{10k+7} = \varepsilon_{10k+8} = \varepsilon_{10k+9} = \varepsilon_{10k+10} = -1$, $k = 0, 1, \ldots, h/10 - 1$ and d, n, h are sufficiently large natural numbers chosen with respect to the restrictions that are introduced in Chapter 7 of [4].

Let G be an arbitrary group; then $w(G)$ denotes the verbal subgroup of G defined by the word $w(x,y)$.

Theorem. *There exists a relatively free group H such that the verbal subgroup $w(H)$ is an infinite cyclic group whereas $w(x,y)$ has only one nontrivial value in H.*

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Proof. Let F_2 be the absolutely free group of rank two. The word $w(x, y)$ coincides with one of the words $w_m(x, y)$, $m = 1, 2, \ldots$, introduced in [5], namely $w_1(x, y)$. Therefore, when studying the group $w(F_2)$ and groups associated with it, we can use lemmas proved in [5]. In particular, by Lemma 17 of [5], all the nontrivial values of $w(x, y)$ in $\overline{G} = F_2/[w(F_2), F_2]$ form a basis of the free abelian group $w(\overline{G})$. Now let V be the subgroup of $\overline{G}$ generated by all the elements of the form $w(X, Y)$ and $w(X_1, Y_1)(w(X_2, Y_2))^{-1}$ for all pairs (X, Y), (X_1, Y_1) and (X_2, Y_2) of words such that the images of the words X and Y do not form a basis of the free abelian group $F_2/[F_2, F_2]$ and the images of the words in each of the pairs (X_1, Y_1) and (X_2, Y_2) form a basis of $F_2/[F_2, F_2]$. By Lemma 10 of [5] and Lemma 15 of [5], the fact that $w(S_1, T_1) = w(S_2, T_2)$ in $\overline{G}$ yields that S_1 is conjugate to S_2 and T_1 is conjugate to T_2 in the group $F_2/w(F_2)$. Hence $S_1 = S_2$ and $T_1 = T_2$ in $F_2/[F_2, F_2]$. Therefore, V is a verbal subgroup of $\overline{G}$ and the group $H = \overline{G}/V$ is a relatively free group such that the image of $w(F_2)$ in H is an infinite cyclic group. It is also clear that there is at most one nontrivial value of the word $w(x, y)$ in H. Thus the Theorem is proved. □

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