PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 124, Number 10, October 1996

SHARP HÖLDER ESTIMATES FOR $\overline{\partial}$ ON ELLIPSOIDS AND THEIR COMPLEMENTS VIA ORDER OF CONTACT

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(Communicated by Eric Bedford)

ABSTRACT. We generalize the results of Range and Diederich et al., finding Hölder estimates for the solution of the Cauchy-Riemann equations for higher order forms on ellipsoids. We prove a dual result near the concave boundaries of complemented complex ellipsoids. In all cases the Hölder exponents are characterized in terms of the order of contact of the boundary of the domain with complex linear spaces of the appropriate dimension. Optimality is demonstrated in the convex settings, and for $(0,1)$ forms in the concave setting. Partial results are given for complemented real ellipsoids and a method for demonstrating optimality of Hortmann's result on complemented strictly pseudoconvex domains is given for $(0,1)$ forms.

INTRODUCTION

Integral operators were used in the early 1970's to yield rather explicit solution operators to the system of inhomogeneous Cauchy-Riemann equations $\overline{\partial}u = f$ on strictly pseudoconvex domains ([Hen] and [GrLi]). These operators were later shown to satisfy sharp Hölder estimates ([Ker], [HeRo], and [RaSi]). Subsequently, similar approaches were used to construct integral solution operators on specific weakly pseudoconvex domains, and modified by Hortmann to treat *complemented strictly pseudoconvex domains* ([Ran1], [DFW], [BrCa], and [Hrt]).

We generalize the results of Range [Ran1] on complex ellipsoids and Diederich, Fornaess, and Wiegerinck [DFW] on real ellipsoids to higher order forms. We show the Hölder exponents obtained are sharp. In addition, these Hölder exponents are of interest in their own right as they are characterized by the reciprocals of the maximal order of contact of the boundaries of the domains with complex linear spaces of the appropriate dimension. This *q-plane type* agrees with other notions of type in the current setting [Yu], thus exhibiting a direct relationship between type and "Hölder gain" for the solution of $\overline{\partial}$ on $(0,q)$ forms on ellipsoids.

We utilize an idea of Hortmann [Hrt] to address the situation near "weakly pseudoconcave" boundaries of *complemented ellipsoids*. Results for forms of all orders are given on *complemented complex ellipsoids*, and optimality is demon-

Received by the editors April 10, 1995.
1991 *Mathematics Subject Classification.* Primary 32F10, 32F20.
This work is part of the authors Ph.D. dissertation at SUNY University at Albany, and was completed while the author was a U.S. Department of Education Fellow.
The author would like to express his gratitude to his advisor Prof. R.M. Range for his continued guidance and support.

strated in the case of $(0,1)$ forms. For *complemented real ellipsoids* the results are only partial. The Hölder exponents obtained in these "weakly pseudoconcave" settings are dual to those in the respective weakly pseudoconvex settings in a natural way.

For $(0,1)$ forms we demonstrate that the result of Hortmann [Hrt] on *complemented strictly pseudoconvex domains* is optimal.

1. Terminology and notation

Let $m = (m_1, ..., m_n) \in \mathbf{N}^n$. We call $D^{(m)} = \left\{ z \in \mathbf{C}^n : \sum_{j=1}^n |z_j|^{2m_j} < 1 \right\}$ a **complex ellipsoid**. Similarly, if $(l,m) = (l_1, ..., l_n, m_1, ..., m_n) \in \mathbf{N}^{2n}$, we call $D^{(l,m)} = \left\{ z = x + iy \in \mathbf{C}^n : \sum_{j=1}^n \left(x_j^{2l_j} + y_j^{2m_j} \right) < 1 \right\}$ a **real ellipsoid**. When they are not balls, ellipsoids of both types are weakly pseudoconvex. Domains of the form $\Omega - \overline{D^{(m)}}$ and $\Omega - \overline{D^{(l,m)}}$, where Ω is a bounded domain, will be called **complemented complex ellipsoids** and **complemented real ellipsoids** respectively. Such domains are "weakly pseudoconcave" near their inner boundaries. We will assume throughout that Ω is a convex domain with C^2 boundary. We are interested in solving the *system of inhomogeneous Cauchy-Riemann equations* $\overline{\partial} u = f$, where the given data f is a $(0,q)$ form with $1 \le q \le n$, with Hölder estimates on complemented and non-complemented complex and real ellipsoids.

All of our solution operators will be constructed via the *Cauchy-Fantappié theory of integral representations* (see e.g. Chapter IV, §3 of [Ran2]). To construct a solution operator to the inhomogeneous Cauchy-Riemann equations on a given domain via this theory, one is required to construct a *generating form* (ibid, p. 169) for the domain whose associated boundary integral in the Cauchy-Fantappié representation vanishes. (More explicitly, with W the generating form, it is required that the integral $\int_{bD} f \wedge \Omega_q(W)$ in (3.14) of [Ran2], Chapter IV, vanishes.)

Notice that from Corollary IV.1.11 of [Ran2] one has a universal kernel that gives a higher dimensional analogue of the Cauchy transform which is a solution operator when $q = n$. From Theorem IV.1.14 of [Ran2] the associated Hölder exponent is independent of the domain in question, so the situation in the $q = n$ case is completely characterized.

2. The convex case

We begin by extending the results of Range [Ran1] and Diederich et al. [DFW] on ellipsoids. On a convex domain D, the **Cauchy form** $C(\xi, z) = \frac{\partial r(\xi)}{\Phi(\xi,z)}$, where r defines D and $\Phi(\xi, z) = \sum_{j=1}^n \frac{\partial r}{\partial \xi_j}(\xi)(\xi_j - z_j)$ is the **linear support function** associated to D at $\xi \in bD$, is a generating form which defines a solution operator. This follows since $\Omega_q(C) = 0$. In fact, whenever $\sum_{j=1}^n P_j(\xi, z)(\xi_j - z_j)$ is a support function and the P_j are holomorphic in the variable z, the form $W(\xi, z) = \frac{\sum_{j=1}^n P_j(\xi,z) d\xi_j}{\sum_{j=1}^n P_j(\xi,z)(\xi_j - z_j)}$ is a generating form which defines a solution operator since one has $\Omega_q(W) = 0$.

Theorem 1. *Let $1 \le q \le n-1$. Let $D = D^{(m)}$ or $D = D^{(l,m)}$ and suppose that $\Delta^q = \Delta^q_{plane}(bD)$ is the maximal order of contact of the boundary of the ellipsoid D with q-dimensional complex linear spaces. Then there are linear operators $T_q : C_{(0,q)}(\overline{D}) \to C_{(0,q-1)}(D)$ satisfying*

i) if $f \in C^1_{(0,q)}(\overline{D})$ and $\overline{\partial} f = 0$, then $\overline{\partial}(T_q f) = f$ on D, and *ii) there is a constant $c < \infty$ such that*

$$|T_q f(z) - T_q f(z')| \le c \, \|f\|_{L^\infty(D)} \, |z - z'|^{\frac{1}{\Delta^q}} \text{ for } z, z' \in D.$$

Remarks. Using the notation in Theorem IV.3.6 of [Ran2] explicit operators satisfying the conclusions of the theorem are T_q^C, where C is the Cauchy form, when $D = D^{(m)}$, and T_q^W, where $W(\xi, z) = \frac{\sum_{j=1}^n P_j(\xi,z) d\xi_j}{\sum_{j=1}^n P_j(\xi,z)(\xi_j - z_j)}$, the P_j as in (3.2) of [DFW], when $D = D^{(l,m)}$. When $q = 1$ these are the operators used by Range and Diederich et al.

If we assume that the coordinates are chosen so that $m_1 \le \cdots \le m_n$ or $min(2l_1, 2m_1) \le \cdots \le min(2l_n, 2m_n)$ respectively, one shows in a straightforward manner that $\Delta^q_{plane}(bD) = 2m_{n-q+1}$ or $min(2l_{n-q+1}, 2m_{n-q+1})$ respectively. So our results agree with those of Range and Diederich et al. in the case $q = 1$.

Proof. When $D = D^{(m)}$ two key features differentiate the $q > 1$ case from the $q = 1$ case treated by Range in [Ran1]. First, Theorem IV.1.14 of [Ran2] suggests that factors in the integral kernel which arises from the *Bochner-Martinelli-Koppelmann* (BMK) form do not play a crucial role in determining the regularity of the solution operator. This is indeed the case. Repeated "non-BMK factors" arise only from the factor $\left(\overline{\partial}_{\xi,\lambda}\hat{C}\right)^{n-q-1}$, so the exponent $n - q - 1$ plays a crucial role in the general setting. (Notice that $n - q - 1 = n - 2$ when $q = 1$, and this is the upper limit on the index s in §5 of [Ran1].) Second, because of these finer restrictions on the index s when $q > 1$ it is useful to introduce a "non-isotropic" version of (5.7) from [Ran1]. Namely, from $(*)$ of [Ran1] one shows

$$(2.1) \qquad r(z) - r(\xi) - 2Re\Phi(\xi, z) \gtrsim \sum_{j=1}^n |\xi_j - z_j|^{2m_j} + \sum_{j=1}^n \frac{\partial^2 r}{\partial \overline{\xi}_j \partial \xi_j}(\xi)|z_j - \xi_j|^2.$$

Using this estimate when $q > 1$ it is possible to pick out smaller exponents when the requisite integrals are estimated. The reader who is interested in working out the details explicitly can follow the example of the proof of Theorem 4 below, incorporating Lemma 4.3 of [DFW] instead of Lemma 5 below.

When $D = D^{(l,m)}$ the situation is similar. The exponent $n - q - 1$ plays a crucial role as the analogue of the upper limit on the index k in (2.1) of [DFW]. As the estimate in Proposition 3.5 of [DFW] is already "non-isotropic," this is the only substantive change. The interested reader is invited to work out the details explicitly, again following the example of the proof of Theorem 4 below and incorporating Lemmas 4.1 and 4.2 of [DFW] instead of Lemma 5 below. □

Remark. The result of McNeal [McN] gives heuristic evidence that the linear support function contains detailed enough information so that the solution operators generated by the Cauchy form "should" resolve the correct Hölder estimates. Efforts on $D^{(l,m)}$ in this direction have only been successful in the $q = n - 1$ case:

Theorem 2. *Let $\Delta^{n-1} = \Delta^{n-1}_{plane}(bD^{(l,m)})$ be the maximal order of contact of the boundary of $D^{(l,m)}$ with $(n-1)$-dimensional complex linear spaces. Then the operator $T^C_{n-1} : C_{(0,n-1)}(\overline{D^{(l,m)}}) \to C_{(0,n-2)}(D^{(l,m)})$ satisfies*

i) if $f \in C^1_{(0,n-1)}(\overline{D^{(l,m)}})$ and $\overline{\partial} f = 0$, then $\overline{\partial}\left(T^C_{n-1} f\right) = f$, and *ii) given* $\epsilon > 0$ *there is a constant* $c_\epsilon < \infty$ *such that*

$$|T^C_{n-1}f(z) - T^C_{n-1}f(z')| \leq c_\epsilon \, \|f\|_{L^\infty(D^{(l,m)})} \, |z - z'|^{\frac{1}{\Delta^{n-1}} - \epsilon} \quad for \quad z, z' \in D^{(l,m)}.$$

Remark. The loss in regularity here is more subtle than the loss in [Ran1], which was corrected in [DFW] by employing additional integral estimates.

Consider the $n = 2, l_1 = 1, m_1 = 2$ case. Using the method below and the obvious estimates, the question of sharpness is tantamount to showing

$$\delta^{\frac{1}{2}} \int_{|s|,|t|,|\lambda| \leq R} \frac{dsdtd\lambda}{\sqrt{s^2 + t^2 + \lambda^2 + \delta^2}\,(\delta + |\lambda| + s^2 + t^4)^2} \leq C < \infty, \tag{2.2}$$

for all $\delta > 0$ sufficiently small. In the complex ellipsoid case, say the ball, the corresponding estimate where t^4 is replaced by t^2 is known to hold. But one shows, by elementary estimates, that (2.2) does not hold.

Proof. As noted in the proof of Theorem 1 above, the exponent $n - q - 1$ plays a crucial role in determining regularity since it gives the number of "non-BMK factors." In this setting there are none as $q = n - 1$. There is a real analogue of (2.1), and this then is all that is required. The interested reader is invited to work out the details explicitly following the example of the proof of Theorem 4 below, using standard estimates instead of Lemma 5 below. □

The results of Theorem 1 when $D = D^{(m)}$ have been shown to be optimal when $q = 1$ ([Ran1], [Kra]) by mimicking the classical example of Stein found in [Ker]. The following modification allows one to treat the cases $q > 1$ as well.

Suppose we have a complex ellipsoid $D^{(m)}$ with $m_1 \leq \cdots \leq m_n$, and $1 \leq q \leq n-1$. Let $\eta(z) = \frac{\overline{z}_n}{log(z_1 - 1)} d\overline{z}_{n-1} \wedge \cdots \wedge d\overline{z}_{n-q+1}$, play the role of v in §2 of [Ran1]. If $\omega \in C^1_{(0,q-1)}(D^{(m)})$ is bounded, $0 < d < \frac{1}{2}$ and we write $0' = \mathbf{0} \in \mathbf{C}^{n-q-2}$, it makes sense to define

$$\int\limits_{bB_d} (\omega(1-d, 0', z_{n-q+1}, ..., z_n) - \omega(1-2d, 0', z_{n-q+1}, ..., z_n)) \wedge dz_n \wedge \cdots \wedge dz_{n-q+1},$$

where $B_d = \left\{ (z_{n-q+1}, ..., z_n) \in \mathbf{C}^q : |(z_{n-q+1}, ..., z_n)| \leq \left(\frac{d}{q}\right)^{\frac{1}{2m_{n-q+1}}} \right\}$. Call this integral $I(d)$, as in [Ran1]. Mimicking Range's subsequent analysis, one obtains:

Lemma 3. *If* $\omega \in C^1_{(0,q-1)}(D^{(m)}) \cap \Lambda_\alpha(D^{(m)})$ *solves* $\overline{\partial}\omega = \overline{\partial}\eta$ *on* $D^{(m)}$, *then* $\alpha \leq \frac{1}{2m_{n-q+1}}$.

As Theorem 1 can be extended, by a limiting argument, to generate Lipschitz solutions when the data is $C^\infty(D^{(m)})$ and $\overline{\partial}$-bounded on $D^{(m)}$, Lemma 3 shows that the Hölder exponent in Theorem 1 is optimal.

To demonstrate optimality for real ellipsoids one simply notes that by defining $\mu_j = min(l_j, m_j)$ one has $D^{(\mu)} \subseteq D^{(l,m)}$ and the orders of contact of the boundaries of $D^{(\mu)}$ and $D^{(l,m)}$ with q-dimensional complex linear spaces are equal. The optimality of Theorem 1 when $D = D^{(l,m)}$ then follows by taking restrictions from $D^{(l,m)}$ to $D^{(m)}$ and employing the reasoning above on $D^{(\mu)}$.

3. The concave case

Suppose that $r \in C^2(\mathbf{C}^n)$ defines a bounded, convex domain D. Let Φ be the associated linear support function as defined above. For $z \notin \overline{D}$ and $\xi \in bD$ we can interchange the roles of the variables in Φ as it is globally defined. This new function, call it $\Phi^{\#}(\xi, z)$, can be realized as the linear support function associated to the convex domain defined by $r - r(z)$ at z evaluated at ξ. By convexity it then follows that $\Phi^{\#}(\xi, z)$ is nonzero on $bD \times \left(\mathbf{C}^n - \overline{D}\right)$. As first noted by Hortmann in [Hrt], this allows us to define a generating form on $R = \Omega - \overline{D}$ when Ω is convex with C^2 boundary and $\overline{D} \subseteq \Omega$. For we can define

$$\mathcal{C}(\xi, z) = \frac{-\partial_\xi r(z)}{\Phi^{\#}(\xi, z)} = \frac{\sum_{j=1}^n \frac{\partial r}{\partial \xi_j}(z) d\xi_j}{\sum_{j=1}^n \frac{\partial r}{\partial \xi_j}(z)(\xi_j - z_j)}$$

on $bD \times \left(\mathbf{C}^n - \overline{D}\right)$ and extend it by the Cauchy form associated to Ω on $b\Omega \times \Omega$. The generating form constructed in this way will be called the **Cauchy-Hortmann form** associated to R.

For $1 \leq q \leq n-2$ this generating form defines a solution operator as its holomorphy properties guarantee that $\Omega_q(W) = 0$. For complemented complex ellipsoids we can obtain Hölder estimates for this solution operator.

Theorem 4. *Let $R = \Omega - \overline{D^{(m)}}$ be a complemented complex ellipsoid; Ω bounded and convex with C^2 boundary. Let $1 \leq q \leq n-2$, $\Delta^{n-q} = \Delta^{n-q}_{plane}(bD^{(m)})$ the maximal order of contact of the boundary of $D^{(m)}$ with $(n-q)$-dimensional complex linear spaces, and $\mathcal{C}$ the Cauchy-Hortmann form associated to R. Then the operators $T_q^{\mathcal{C}} : C_{(0,q)}(\overline{R}) \to C_{(0,q-1)}(R)$ satisfy*

i) if $f \in C^1_{(0,q)}(\overline{R})$ and $\overline{\partial} f = 0$, then $\overline{\partial}\left(T_q^{\mathcal{C}} f\right) = f$, and

ii) there are $\delta > 0$ and $c < \infty$ so that if $z, z' \in R$ and $\delta_{bD^{(m)}}(z), \delta_{bD^{(m)}}(z') < \delta$, then

$$|T_q^{\mathcal{C}} f(z) - T_q^{\mathcal{C}} f(z')| \leq c \, \|f\|_{L^\infty(R)} \, |z - z'|^{\frac{1}{\Delta^{n-q}}}.$$

Proof. By Theorem IV.1.14 of [Ran2] the regularity of the solution operator is determined by the boundary integral in the definition of $T_q^{\mathcal{C}}$ alone. In addition, as we are only concerned with the situation near the inner boundary, the outer boundary integral will be inconsequential to the estimate in ii). We will use a gradient estimate (Lemma V.3.1 of [Ran2]) to ascertain the regularity of the integral in question.

Our homotopy form on $(bD^{(m)} \times I) \times (\Omega - \overline{D^{(m)}})$ is given by $\hat{\mathcal{C}}(\xi, \lambda, z) = \lambda \mathcal{C}(\xi, z) + (1-\lambda)\frac{\partial_\xi |\xi - z|^2}{|\xi - z|^2}$, where $\mathcal{C}$ is as above. As $\overline{\partial}_\xi \mathcal{C}(\xi, z), \mathcal{C}(\xi, z) \wedge \mathcal{C}(\xi, z), \partial_\xi |\xi - z|^2 \wedge \partial_\xi |\xi - z|^2 = 0$, the kernel in question, $\Omega_{q-1}(\hat{\mathcal{C}}) = c_{n,q} \hat{\mathcal{C}} \wedge \left(\overline{\partial}_{\xi,\lambda} \hat{\mathcal{C}}\right)^{n-q} \wedge \left(\overline{\partial}_z \hat{\mathcal{C}}\right)^{q-1}$, simplifies to

$$\Omega_{q-1}(\hat{\mathcal{C}})(\xi, \lambda, z) = \omega(\xi, \lambda, z) + \sum_{k=0}^{q-1} p_k(\lambda) d\lambda \wedge \mathcal{C}(\xi, z) \wedge \frac{\partial_\xi |\xi - z|^2}{|\xi - z|^2} \wedge \left(\overline{\partial}_\xi \left(\frac{\partial_\xi |\xi - z|^2}{|\xi - z|^2}\right)\right)^{n-q-1} \wedge \left(\overline{\partial}_z \mathcal{C}(\xi, z)\right)^k \wedge \left(\overline{\partial}_z \left(\frac{\partial_\xi |\xi - z|^2}{|\xi - z|^2}\right)\right)^{q-k-1},$$

where the p_k are polynomials in λ, and ω is a 0-form in λ. The form ω integrates to 0 over the boundary in question and the homotopy variable λ integrates simply.

Differentiating and applying $\partial_\xi r(z) \wedge \partial_\xi r(z), \partial_\xi |\xi - z|^2 \wedge \partial_\xi |\xi - z|^2 = 0$ simplifies the resulting kernel to a sum of terms of the form

$$\frac{\partial_\xi r(z)}{\Phi^\#(\xi, z)} \wedge \frac{\partial_\xi |\xi - z|^2}{|\xi - z|^2} \wedge \left[\frac{\overline{\partial}_\xi \partial_\xi |\xi - z|^2}{|\xi - z|^2}\right]^{n-q-1} \wedge \left[\frac{\overline{\partial}_z \partial_\xi r(z)}{\Phi^\#(\xi, z)}\right]^k \wedge \left[\frac{\overline{\partial}_z \partial_\xi |\xi - z|^2}{|\xi - z|^2}\right]^{q-k-1},$$

where $0 \le k \le q - 1$.

We would like to estimate the absolute value of terms in the gradient of the kernel. Most estimates will be straightforward, e.g. $\left\|\partial_\xi |\xi - z|^2\right\| \le |\xi - z|$, $\left\|\overline{\partial}_z \partial_\xi |\xi - z|^2\right\| \le 1$, but factors of the form $\partial_\xi r(z) \wedge \left(\overline{\partial}_z \partial_\xi r(z)\right)^k$ must be treated delicately. As the defining function satisfies $\frac{\partial^2 r}{\partial \overline{\xi}_j \partial \xi_k} = 0$ if $j \ne k$, the factors in question can be expanded into sums of terms of the form

$$\left(\frac{\partial r}{\partial \xi_{q_0}}(z)\right) \left[\prod_{j=1}^k \left(\frac{\partial^2 r}{\partial \overline{\xi}_{q_j} \partial \xi_{q_j}}(z)\right)\right] d\xi_{q_0} \wedge d\overline{z}_{q_1} \wedge d\xi_{q_1} \wedge \cdots \wedge d\overline{z}_{q_k} \wedge d\xi_{q_k}, \tag{3.1}$$

where the q_j are distinct. Again, by the special nature of the defining function, $\left|\frac{\partial r}{\partial \xi_{q_0}}(z)\right| \le \left|\frac{\partial^2 r}{\partial \overline{\xi}_{q_0} \partial \xi_{q_o}}(z)\right|$, so the term in (3.1) can be estimated above by a constant multiple of $\prod_{j=0}^k \left|\frac{\partial^2 r}{\partial \overline{\xi}_{q_j} \partial \xi_{q_j}}(z)\right|$, where the q_j are distinct. Hence, any term in the gradient of the kernel is bounded above in absolute value by a constant multiple of one of

(3.2)

$$\frac{|\xi - z| \prod_{j=0}^k \left|\frac{\partial^2 r}{\partial \overline{\xi}_{q_j} \partial \xi_{q_j}}(z)\right|}{|\Phi^\#|^{k+2} |\xi - z|^{2(n-k-1)}}, \quad \frac{|\xi - z|^2 \prod_{j=0}^k \left|\frac{\partial^2 r}{\partial \overline{\xi}_{q_j} \partial \xi_{q_j}}(z)\right|}{|\Phi^\#|^{k+1} |\xi - z|^{2(n-k)}}, \quad \frac{|\xi - z| \prod_{j=0}^{k-1} \left|\frac{\partial^2 r}{\partial \overline{\xi}_{q_j} \partial \xi_{q_j}}(z)\right|}{|\Phi^\#|^{k+1} |\xi - z|^{2(n-k-1)}},$$

where the q_j are distinct.

We will parameterize the boundary integral locally as follows. (Compare with that given in §5 of [Ran1].) Near each boundary point z of $D^{(m)}$ some $z_j \ne 0$; say $z_n \ne 0$ for concreteness. Given $0 < \eta, \delta < \infty$ there are positive constants R and γ, depending on η and δ, such that for $\delta_{bD}(z) \le \delta$ with $\left|\frac{\partial r}{\partial z_n}(z)\right| \ge \eta$ the system

$$\begin{aligned} s_j &= Re(\xi_j - z_j), \ for\ 1 \le j \le n-1, \\ t_j &= Im(\xi_j - z_j), \ for\ 1 \le j \le n-1, \\ \lambda &= Im\Phi^\#(\xi, z), \\ \rho &= r(\xi) + |r(z)| \end{aligned}$$

defines a C^1 coordinate system $(s_1, t_1, ..., s_{n-1}, t_{n-1}, \lambda, \rho)$ on $B_\gamma(z)$, which is uniform in z and which satisfies

(i) $bD \cap B_\gamma(z) \subseteq \{(s_1, t_1, ..., s_{n-1}, t_{n-1}, \lambda, \rho) : |s|, |t|, |\lambda| \le R, \rho = |r(z)|\}$, and

(ii) $\left\| d\sigma|_{bD \cap B_\gamma(z)} \right\| \lesssim \|ds_1 \wedge dt_1 \wedge \cdots \wedge ds_{n-1} \wedge dt_{n-1} \wedge d\lambda\|$,

uniformly in z.

Applying this parameterization and the estimate in (2.1), with the roles of the variables ξ, z interchanged, gives the following partial bound for the integral gradient estimate in question:

$$\int\limits_{bD^{(m)}\cap B_\gamma(z)} \frac{|\xi - z| \prod_{j=1}^{k} \left|\frac{\partial^2 r}{\partial\bar\xi_{n-j}\partial\xi_{n-j}}(z)\right| dS(\xi)}{|\Phi^{\#}(\xi,z)|^{k+2}|\xi - z|^{2(n-k-1)}}$$

$$\lesssim \int\limits_{0\le\lambda,s_j,t_j\le R} \frac{\prod_{j=1}^{k}\left|\frac{\partial^2 r}{\partial\bar\xi_{n-j}\partial\xi_{n-j}}(z)\right|}{\left(|r(z)| + \lambda + \sum_{j=1}^{n-1}\left(\left|\frac{\partial^2 r}{\partial\bar\xi_j\partial\xi_j}(z)\right|(s_j^2+t_j^2) + (s_j^2+t_j^2)^{m_j}\right)\right)^{k+2}}$$
$$\times \frac{ds_1dt_1\cdots ds_{n-1}dt_{n-1}d\lambda}{(s_1+...+t_{n-1}+\lambda+|r(z)|)^{2(n-k-1)-1}}.$$

Notice we have replaced the $(k+1)$-fold product by a k-fold product as the case when some $q_j = n$ will be of little use to us. Additionally, we have tacitly assumed that $p_j = n-j$ for each j. Deleting λ from the second factor in the denominator we can integrate in λ reducing the exponent $k+2$ to $k+1$. The following elementary lemma will be used to estimate the remaining integral:

Lemma 5. *Let $A, C \ge 0$ be parameters, $0 < M, R < \infty$ fixed constants with $C \le M$ for all values of the parameters in question. Then for $q \ge 1$*

$$\int\limits_{0\le|(s,t)|\le R} \frac{C\,ds\,dt}{(A+C(s^2+t^2))^q} \lesssim \begin{cases} |ln(A)| & \text{if } q=1,\\ \frac{1}{A^{q-1}} & \text{if } q>1,\end{cases}$$

uniformly in C for A sufficiently small.

Drop $(s_{n-1}^2+t_{n-1}^2)^{m_{n-1}}$ from the first factor in the denominator, s_{n-1}, t_{n-1} from the second factor in the denominator and integrate using Lemma 5. Repeating this process $k-1$ more times and then deleting the remaining terms $\left|\frac{\partial^2 r}{\partial\bar z_{n-j}\partial z_{n-j}}(z)\right|(s_{n-j}^2+t_{n-j}^2)$ in the first factor in the resulting denominator, we are left with the following bound for the integral above:

$$\int\limits_{0\le s_j,t_j\le R} \frac{ds_1\cdots dt_{n-k-1}}{\left(|r(z)| + \sum_{j=1}^{n-k-1}(s_j^2+t_j^2)^{m_j}\right)(s_1+...+t_{n-k-1}+|r(z)|)^{2(n-k-1)-1}}.$$

Replacing $\sum_{j=1}^{n-k-1}\left(s_j^2+t_j^2\right)^{m_j}$ by $\left(s_1^2+t_1^2\right)^{m_1}$ increases the bound. Successive integration in the variables $s_2, t_2, ..., s_{n-k-1}, t_{n-k-1}$ then yields the upper bound

$$\int\limits_{0\le s_1,t_1\le R} \frac{ds_1dt_1}{(|r(z)|+(s_1^2+t_1^2)^{m_1})(s_1+t_1+|r(z)|)}.$$

Introducing polar coordinates and the estimate $\frac{r}{r+|r(z)|}\le 1$, we are left to investigate $\int_{0\le r\le R}\frac{dr}{|r(z)|+r^{2m_1}}$. This is easily shown to be bounded by a constant multiple of $|r(z)|^{\frac{1}{2m_1}-1}$.

The estimate involving the third term in (3.2) is handled in an analogous way, with the same result. Estimates involving the second term in (3.2) follow from our work above as $|\Phi^{\#}(\xi, z)| \underset{\sim}{<} |\xi - z|$.

Notice there is nothing special about the exponent m_1, any of the other exponents $m_2, ..., m_{n-k-1}$ could have played the central role. Since it is possible we had to interchange coordinates so that $z_n \neq 0$ and $p_j = n - j$ for $1 \leq j \leq k$, globally we are only free to choose as our exponent the minimum of some collection $n - (k+1)$ members of $\{2m_j\}$. As k is at most $q - 1$ one sees readily that the best exponent is the maximal order of contact of $bD^{(m)}$ with $(n - q)$-dimensional complex linear spaces. Since $|r(z)| \sim \delta_{bD^{(m)}}(z)$, this gives the desired gradient estimate and the result follows. □

When $q = n-1$ the kernel $\Omega_q(\mathcal{C})$ does not necessarily vanish, reflecting the nonvanishing of the cohomology class $H_{\overline{\partial}}^{(0,n-1)}(R)$. However, if the data f is known to be $\overline{\partial}$-exact, one uses Stokes' theorem to show that the boundary integral $\int_{bR} f \wedge \Omega_{n-1}(\mathcal{C})$ vanishes even though the kernel does not. In that case the analysis above can be repeated, and we obtain:

Theorem 6. *In the setting of Theorem 4, there are $\delta > 0$ and $c < \infty$ so that the operator $T_{n-1}^{\mathcal{C}} : C_{(0,n-1)}(\overline{R}) \to C_{(0,n-2)}(R)$ satisfies:*
i) $\overline{\partial}\left(T_{n-1}^{\mathcal{C}} f\right) = f$, and
ii) if $z, z' \in R$ and $\delta_{bD^{(m)}}(z), \delta_{bD^{(m)}}(z') < \delta$, then

$$\left|T_{n-1}^{\mathcal{C}} f(z) - T_{n-1}^{\mathcal{C}} f(z')\right| \leq c \, \|f\|_{L^\infty(R)} \, |z - z'|^{\frac{1}{\Delta^1}}$$

whenever
a) $f \in C^1_{(0,n-1)}(\overline{R})$ and $\overline{\partial} f = 0$ on R,
b) f is $\overline{\partial}$ exact in $U \cap R$ for some neighborhood U of $bD^{(m)}$.

We would like to show that the result of Theorem 4 is optimal. Returning to the convex setting, in the definition of η the function $log(z_1 - 1)$, or simply $z_1 - 1$, played a crucial role in our demonstration of optimality. This is just a multiple of the linear support function $\Phi(e_1, z)$. In the concave setting it is natural to investigate $\Phi^{\#}(\xi_0, z)$ for some $\xi_0 \in bD$. Only now the variable z does not appear linearly! This procludes us from mimicking the proof of Lemma 3. In particular, using Stokes' theorem to gain information about any solution from a specific solution will not generally be possible. In the concave case when $q = 1$ this drawback can be remedied by employing Hartog's extension theorem. Namely,

Lemma 7. *Let $D \subseteq \overline{D} \subseteq \Omega \subseteq \mathbf{C}^n$, $n > 1$, be bounded domains with $R = \Omega - \overline{D}$ connected. If $u, v \in C^1(R)$ satisfy $\overline{\partial} u = \overline{\partial} v$ on R, then $u - v \in C^\infty(\Omega)$.*

Thus, when $q = 1$, any two solutions u of $\overline{\partial} u = f$ have the same regularity near the inner boundary. The following theorem therefore proves the optimality of the estimate in Theorem 4 for $q = 1$:

Theorem 8. *Suppose $R = \Omega - \overline{D^{(m)}}$ is a complemented complex ellipsoid. Suppose $m_n \leq m_1 \leq m_2, ..., m_{n-1}$ so that $2m_1 = \Delta^{n-1} = \Delta^{n-1}_{plane}(bD^{(m)})$, the maximal order of contact of the boundary of $D^{(m)}$ with $(n-1)$-dimensional complex linear spaces. Let $\Phi(\xi, z)$ be the linear support function associated to $D^{(m)}$ at $\xi \in bD^{(m)}$ as usual, $\Phi^{\#}(\xi, z) = \Phi(z, \xi)$ as above, and $\xi_0 = (0, ..., 0, 1)$. Then the function v defined on*

$\mathbf{C}^n - \overline{D^{(m)}}$ by

$$v(z) = \frac{z_1^{2m_1+1}}{\Phi^{\#}(\xi_0, z)} = \frac{z_1^{2m_1+1}}{m_1|z_1|^{2m_1} + ... + m_n|z_n|^{2m_n} - m_n\overline{z}_n|z_n|^{2m_n-2}}$$

is a well defined C^∞ function satisfying:
i) $\overline{\partial}v \in L^\infty(R)$,
ii) $v \in \Lambda_{\frac{1}{\Delta^{n-1}}}(R)$, and
iii) $v \notin \Lambda_\alpha(R)$ for any $\alpha > \frac{1}{\Delta^{n-1}}$.

Proof. We give an outline; details may be found in [Fle]. One proves i) by lengthy, delicate estimations, relying heavily on (2.1). Similar estimates can be used to show $\|\partial v(z)\| \lesssim \delta_{bD^{(m)}}(z)^{\frac{1}{2m_1}-1}$. Then ii) follows from i) and Lemma IV.3.1 of [Ran2]. For iii) note that if $0' = \mathbf{0} \in \mathbf{C}^{n-2}$, then $\left(z_1, 0', 1 - \frac{\delta^{2m_1}}{4m_n}\right), \left(z_1, 0', 1 - \frac{\delta^{2m_1}}{2m_n}\right) \notin \overline{D^{(m)}}$ when $|z_1| = \delta$. Hence, for $\delta > 0$ sufficiently small it makes sense to define

$$I(\delta) = \int_{|z_1|=\delta, 0\leq Arg(z_1)\leq\frac{\pi}{2m_1+2}} \left[v\left(z_1, 0', 1 - \frac{\delta^{2m_1}}{4m_n}\right) - v\left(z_1, 0', 1 - \frac{\delta^{2m_1}}{2m_n}\right)\right] dz_1.$$

As in [Ran1], one can evaluate $I(\delta)$ directly or one can estimate the integral using the fact that $v \in \Lambda_\alpha(R)$, arriving at incompatible results when $2m_1\alpha > 1$. □

Remarks. Unlike the convex setting, this result does not immediately give the corresponding result about complemented real ellipsoids. It is feasible the analogous construction would give such a result.

There is a related construction subsequently suggested by R.M. Range which addresses the complemented strictly pseudoconvex case treated in [Hrt]. Namely, suppose $R = \Omega - \overline{D}$, where D is strictly pseudoconvex, $\overline{D} \subseteq \Omega$, and $\Omega - \overline{D}$ is connected. One shows that near a fixed boundary point ξ_0 of bD, possibly after a holomorphic change of coordinates, there is a defining function ρ for D which satisfies

$$-2Re(\Phi(\xi, z)) \gtrsim \rho(\xi) - \rho(z) + |\xi - z|^2 \tag{3.3}$$

for all ξ, z sufficiently close to ξ_0, where Φ is the linear support function as before. If $\Phi^{\#}(\xi, z) = \Phi(z, \xi)$, as above, then using (3.3) one shows that there is a function $v \in C^\infty(\mathbf{C}^n - \overline{D})$ which is given locally by $v(z) = \sqrt{-2\Phi^{\#}(\xi_0, z)}$, and which satisfies

$$\overline{\partial}v \in L^\infty(\mathbf{C}^n - \overline{D}), v \in \Lambda_{\frac{1}{2}}(R), \text{ and } v \notin \Lambda_\alpha(R) \text{ for any } \alpha > \frac{1}{2}.$$

Combinded with Lemma 7 this shows that on any complemented strictly pseudoconvex domain the result of Hortmann is optimal.

We have yet to discuss complemented real ellipsoids. One expects that an analogue of Theorem 4 would hold, with the possibility that the generating form is the "DFW-Hortmann form," that is, the form defined by switching the roles of the variables in the higher order support function of Diederich et al. used in Theorem 1. Although a solution operator is defined in this way, Hölder estimates do not seem to follow as expected. The central problem is that by interchanging the roles of the variables the crucial vanishing in the numerator of the kernel depends on the parameter z and not on a variable of integration. There do not appear to be analogues of Lemmas 4.1 and 4.2 of [DFW] to handle this situation.

At this time we can show an analogue of Theorem 2.

Theorem 9. *Let $R = \Omega - \overline{D^{(l,m)}} \subseteq \mathbf{C}^n$, with $n \geq 3$, be a complemented real ellipsoid; Ω bounded and convex with C^2 boundary. Let $\mathcal{C}$ be the Cauchy-Hortmann form associated to R and let $\Delta^{n-1} = \Delta^{n-1}_{plane}(bD^{(l,m)})$ be the maximal order of contact of the boundary of $D^{(l,m)}$ with $(n-1)$-dimensional complex linear spaces. Then the operator $T_1^{\mathcal{C}} : C_{(0,1)}(\overline{R}) \to C(R)$ satisfies*

i) if $f \in C^1_{(0,1)}(\overline{R})$ and $\overline{\partial} f = 0$, then $\overline{\partial}\left(T_1^{\mathcal{C}} f\right) = f$, and

ii) there are $\delta > 0$ and $c_\epsilon < \infty$ so that if $z, z' \in R$ and $\delta_{bD^{(l,m)}}(z), \delta_{bD^{(l,m)}}(z') < \delta$, then

$$|T_1^{\mathcal{C}} f(z) - T_1^{\mathcal{C}} f(z')| \leq c_\epsilon \, \|f\|_{L^\infty(R)} \, |z - z'|^{\frac{1}{\Delta^{n-1}} - \epsilon}.$$

Remark. The loss of regularity here is analogous to that in Theorem 2.

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