

**MULTIPLE GEOMETRICALLY DISTINCT CLOSED
NONCOLLISION ORBITS OF FIXED ENERGY
FOR N-BODY TYPE PROBLEMS
WITH STRONG FORCE POTENTIALS**

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ABSTRACT. Using the equivariant Ljusternik-Schnirelmann theory, we prove that there are at least $2(N-1)2^{N-2}$ geometrically distinct noncollision orbits with prescribed energy for a class of planar N-body type problems with strong force potentials.

1. INTRODUCTION

In the last few years, variational methods have started to be used for studying the periodic solutions for singular Hamiltonian systems. Existence of periodic solutions with prescribed period for some classes of N-body problems has been proved in [2]–[6], [9], [11]. On the contrary, we know of only one paper ([1]) obtaining the results in the large concerning the existence of one periodic solution with prescribed energy, and except for the author's preprint paper ([14]), there is no result in the large dealing with the existence of multiple geometrically distinct closed noncollision orbits. This is mainly because the N-body problems in \mathbb{R}^k have $S^1 \times O(k)$ symmetry, where $O(k)$ is the rotational symmetry group of order k . In order to obtain multiple geometrically distinct trajectories, we must consider the effects of the group $S^1 \times O(k)$; this results in some problems in topology.

Using the equivariant Ljusternik-Schnirelmann theory, we obtain the result in the large concerning multiple geometrically distinct closed noncollision orbits for a class of planar N-body problems with strong force potentials. The key observations of the proof about our results are that the variational functional satisfies the Palais-Smale condition and that we must construct a special set such that its $S^1 \times O(2)$ equivariant category is easily computed and the superbound of the functional on the set is also easily estimated.

Many variational methods' papers deal with N-body type problems, but assume conditions on the potential that exclude the actual N-body problem (i.e. gravitational potential). This paper is no exception, but in another paper ([14]), the author does deal with gravitational potentials.

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We set $\Omega = \mathbb{R}^2 \setminus \{0\}$ and consider a potential V of the form:

$$(1.1) \quad V(x) = V(x_1, \dots, x_N) = \frac{1}{2} \sum_{1 \leq i \neq j \leq N} V_{ij}(x_i - x_j),$$

where $x_i \in \mathbb{R}^2$, $x = (x_1, \dots, x_N) \in (\mathbb{R}^2)^N$ and $V_{ij} \in C^1(\Omega, \mathbb{R})$ ($i, j = 1, \dots, N$).

Given $m_i > 0$ ($i = 1, \dots, N$) and $h \in \mathbb{R}$, we seek periodic solutions of

$$(Ph) \quad \begin{cases} m_i \ddot{x}_i + V'_{x_i}(x_1, \dots, x_N) = 0, & 1 \leq i \leq N, \\ \frac{1}{2} \sum_{i=1}^N m_i |\dot{x}_i|^2 + V(x_1, \dots, x_N) = h, & 1 \leq i \leq N. \end{cases}$$

Definition 1.1. If the solution $x = (x_1, \dots, x_N)$ of (Ph) satisfies

- (i) $x_i \in C^2(0, T; \mathbb{R}^2)$,
- (ii) $x_i(t) \neq x_j(t), \forall t \in [0, T]$ and $\forall 1 \leq i \neq j \leq N$,

then we call x the noncollision periodic solution of (Ph).

Definition 1.2. Let x and y be two periodic solutions of (Ph), neither of which can be brought into the other by the standard $S^1 \times O(2)$ action. Then we call x and y distinct in geometrics.

Theorem 1.3. Assume V possesses the form of (1.1) and V_{ij} and V satisfy

(V1) $V_{ij}(\xi) = V_{ji}(\xi), \forall i \neq j, \forall \xi \in \mathbb{R}^2$, and $V(u_1, \dots, u_N) = V(Ru_1, \dots, Ru_N), \forall R \in O(2), \forall (u_1, \dots, u_N) \in (\mathbb{R}^2)^N$.

(V2) There are $\alpha > 2$ and $a > b > 0$ such that for any $x_i \in \mathbb{R}^2$ and $x_i \neq x_j$

$$(1.2) \quad -\frac{a}{2} \sum_{1 \leq i \neq j \leq N} \frac{m_i m_j}{|x_i - x_j|^\alpha} \leq V(x_1, \dots, x_N) \leq -\frac{b}{2} \sum_{i \neq j} \frac{m_i m_j}{|x_i - x_j|^\alpha}.$$

$$(V3) \quad c_2 > \frac{1}{9} c_5,$$

where

$$\begin{aligned} c_5 &= \frac{1}{2} c_4^{2/\alpha} \left(\frac{\alpha - 2}{2} \right)^{2/\alpha} \left(c_4 + \frac{2}{\alpha - 2} c_3 \right), \\ c_4 &= \sum_{i=1}^N 4\pi^2 i^2 m_i, \\ c_3 &= \frac{a}{2} \sum_{i \neq j} \frac{m_i m_j}{|i - j|^\alpha}, \\ c_2 &= \frac{2\pi^2 \alpha}{\alpha - 2} \left(\frac{\alpha - 2}{2} \right)^{2/\alpha} (bc_1)^{2/\alpha}, \\ c_1 &= 2^{-(\alpha+2)/2} \left(\sum_{i \neq j} m_i m_j \right)^{(\alpha+2)/2} m^{-\alpha/2}, \\ m &= \sum_{i=1}^N m_i. \end{aligned}$$

Then for any $h > 0$, (Ph) has at least $2(N - 1)2^{N-2}$ geometrically distinct noncollision periodic solutions.

Corollary 1.4. *In Theorem 1.3, let $N = 3$, $m_1 = 1$, $m_2 = \varepsilon \ll 1$, $m_3 = 1/4$, $\alpha \rightarrow 2$, and $a = b = 1$. Then for any $h > 0$, (Ph) has at least 8 geometrically distinct noncollision periodic solutions.*

Proof. Under the assumption of Corollary 1.4, we have

$$m \approx 5/4, \quad c_1 \approx 1/20, \quad c_2 \approx \pi^2/10, \quad c_3 \approx 1/16, \\ c_4 \approx 13\pi^2, \quad c_5 \approx (1/2)c_3 \times c_4 \approx (13/32)\pi^2 < 9c_2.$$

2. FUNCTIONAL SETTING AND THE PROOF OF THEOREM 1.3

Let us introduce the following notation:

$$H = W^{1,2}(S^1, \mathbb{R}^2), \\ H_{\#} = \left\{ u \in H \mid u\left(t + \frac{1}{2}\right) = -u(t) \right\}, \\ E = \{u = (u_1, \dots, u_N) \mid u_i \in H_{\#}, i = 1, \dots, N\}, \\ \Lambda_0 = \{u \in E \mid u_i(t) \neq u_j(t), \forall t, i \neq j\}, \\ \langle u, v \rangle = \int_0^1 \dot{u} \dot{v} dt, \quad \|u\|^2 = \int_0^1 |\dot{u}|^2 dt, \quad \forall u, v \in H_{\#}.$$

It is well known that $\|u_i\|$ is a norm which is equivalent to the usual one and we have

$$\|u_i\| \geq 4|u_i|_{\infty}.$$

Hence for all $u = (u_1, \dots, u_N) \in E$, setting

$$\|u\|_E^2 = \sum_{i=1}^N m_i \|u_i\|^2,$$

we obtain

$$(2.1) \quad \|u\|_E^2 \geq 16\underline{m}|u|_{\infty}^2,$$

where $\underline{m} = \min\{m_i \mid i = 1, \dots, N\}$ and $|u|_{\infty}^2 = \sum_{i=1}^N |u_i|_{\infty}^2$.

On Λ_0 define the following functional:

$$(2.2) \quad f(u) = \frac{1}{2} \|u\|_E^2 \int_0^1 (h - V(u)) dt.$$

Similar to the proof of [1] and [8], we have the following variational principle:

Lemma 2.1. *Let $u \in \Lambda_0$ be the critical point of f and $\|u\|_E > 0$. Let*

$$(2.3) \quad \omega^2 = \frac{\int_0^1 V'(u) \cdot u dt}{\|u\|_E^2} > 0.$$

Then $x(t) = u(\omega t)$ is a noncollision periodic solution of (Ph).

Lemma 2.2 ([5]). *Let $X = (x_1, \dots, x_N) \in (\mathbb{R}^k)^N$. Then*

$$(2.4) \quad \frac{1}{2} \sum_{1 \leq i \neq j \leq N} \frac{m_i m_j}{|x_i - x_j|^\alpha} \geq c_1 \frac{1}{(\sum_{i=1}^N m_i |x_i|^2)^{\alpha/2}}.$$

Lemma 2.3. *Suppose V satisfies (V2) of Theorem 1.3. Then*

$$(2.5) \quad -V(x) \geq c \frac{1}{(\sum_{i=1}^N |x_i|^2)^{\alpha/2}}.$$

Proof. By $\sum_{i=1}^N m_i |x_i|^2 \leq \bar{m} \sum_{i=1}^N |x_i|^2$ and Lemma 2.2, the inequality (2.5) is proved.

Lemma 2.4. *Assume (V1) and (V2) hold. Then $f \in C^1(\Lambda_0, \mathbb{R})$ and*

$$(2.6) \quad f(u_n) \rightarrow +\infty, \quad \forall u_n \rightharpoonup u \in \partial\Lambda.$$

Proof. (i) If $u(t) \equiv 0$, then by the embedding theorem,

$$(2.7) \quad |u_n|_\infty \rightarrow 0, \quad n \rightarrow \infty.$$

Hence by inequality (2.1), (V2) and Lemma 2.3, we have

$$\begin{aligned} f(u_n) &= \frac{1}{2} \|u_n\|_E^2 \int_0^1 (h - V(u_n)) dt \geq \frac{1}{2} \|u_n\|_E^2 \int_0^1 (-V(u_n)) dt \\ &\geq 8\underline{m} |u_n|_\infty^2 c \int_0^1 |u_n|^{-\alpha} dt \geq 8\underline{m} c |u_n|_\infty^2 |u_n|_\infty^{-\alpha}. \end{aligned}$$

Hence $f(u_n) \rightarrow +\infty$ as $n \rightarrow \infty$.

(ii) If $u(t) \not\equiv 0$, then $\|u\|_E^2 = \sum_{i=1}^N m_i \|u_i\|_2^2 \neq 0$. In fact, if $\|u\|_E = 0$, then $u_i(t) \equiv c = 0$ by $u_i(t + \frac{1}{2}) = -u_i(t)$. By the weak lower semi-continuous property of $g(u) = \|u\|_E$, we have

$$(2.8) \quad \liminf_{n \rightarrow \infty} \|u_n\|_E \geq \|u\|_E > 0.$$

By (V2), V_{ij} satisfies the strong force condition of Gordon ([7]), and so

$$(2.9) \quad \int_0^1 V(u_n) dt \rightarrow -\infty, \quad \forall u_n \rightharpoonup u \in \partial\Lambda_0.$$

Hence by (2.8) and (2.9), we have

$$f(u_n) = \frac{1}{2} \|u\|_E^2 \int_0^1 (h - V(u_n)) dt \rightarrow +\infty, \quad n \rightarrow \infty.$$

Lemma 2.5. *f satisfies the Palais-Smale condition in Λ_0 .*

Proof. Assume $\{u_n\} \subset \Lambda_0$ and that it satisfies

$$(2.10) \quad |f(u_n)| \leq M,$$

$$(2.11) \quad f'(u_n) \rightarrow 0.$$

Then by (2.10) and $V(u_n) \leq 0$, we know

$$0 \leq -\frac{1}{2} \|u_n\|_E^2 \int_0^1 V(u_n) dt \leq M - \frac{h}{2} \|u_n\|_E^2;$$

hence $\|u_n\|$ is bounded, and there is a weakly convergent subsequence $\{u_{n_k}\}$ of $\{u_n\}$. Similar to the proof of [1], $u_{n_k} \rightharpoonup u \in \Lambda_0$, and hence f satisfies the Palais-Smale condition.

Lemma 2.6. *On Λ_0 , f possesses a positive lower bound:*

$$\inf_{u \in \Lambda_0} f(u) \geq c_2.$$

Proof. By Lemma 2.2, for any $u \in \Lambda_0$, we have that

$$f(u) \geq \frac{1}{2} \int_0^1 \left(\sum_{i=1}^N m_i |\dot{u}_i|^2 \right) dt \left(h + bc_1 \int_0^1 \left(\sum_{i=1}^N m_i |u_i|^2 \right)^{-\alpha/2} dt \right).$$

By the Hölder inequality, we have

$$\int_0^T |q| dt \leq T^{1/2} \left(\int_0^T |q|^2 dt \right)^{1/2}, \quad \left(\int_0^T |q|^2 dt \right)^{\alpha/2} \geq T^{-\alpha/2} \left(\int_0^T |q| dt \right)^\alpha.$$

By the Chebychev inequality, we have

$$\int_0^T \frac{1}{|q|} dt \cdot \int_0^T |q| dt \geq T^2,$$

so

$$\left(\int_0^T \frac{1}{|q|} dt \right)^\alpha \geq T^{3\alpha/2} \left(\int_0^T |q|^2 dt \right)^{-\alpha/2}.$$

Using the above results and the Jensen inequality, we have

$$\begin{aligned} \int_0^T \frac{1}{|q|^\alpha} dt &\geq T \left(\frac{1}{T} \int_0^T \frac{1}{|q|} dt \right)^\alpha \geq T^{1+\alpha/2} \left(\int_0^T |q|^2 dt \right)^{-\alpha/2}, \\ \int_0^T \left(\sum_{i=1}^N m_i |u_i|^2 \right)^{-\alpha/2} dt &\geq T^{1+\alpha/2} \left[\int_0^T \left(\sum_{i=1}^N m_i |u_i|^2 \right) dt \right]^{-\alpha/2}. \end{aligned}$$

By the Wirtinger inequality, we have that

$$f(u) \geq 2\pi^2 \int_0^1 \left(\sum_{i=1}^N m_i |u_i|^2 \right) dt \left\{ h + bc_1 \left[\int_0^1 \left(\sum_{i=1}^N m_i |u_i|^2 \right) dt \right]^{-\alpha/2} \right\}.$$

Let $r^2 = \int_0^1 (\sum_{i=1}^N m_i |u_i|^2) dt$ and $g(r) = 2\pi^2 r^2 (h + bc_1 r^{-\alpha})$. Then $f(u) \geq \min_{r>0} g(r)$ since $r = r_0 = [(\alpha - 2)bc_1/2h]^{1/\alpha}$ satisfies $g'(r_0) = 0$ and $g''(r_0) > 0$. Hence $g(r)$ has minimum value $g(r_0)$ and

$$f(u) \geq g(r_0) = \frac{2\pi^2\alpha}{\alpha - 2} \left(\frac{\alpha - 2}{2} \right)^{2/\alpha} (bc_1)^{2/\alpha} \cdot h^{(\alpha-2)/\alpha} \equiv c_2 h^{(\alpha-2)/\alpha}.$$

In the following, we define a $S^1 \times O(2)$ invariant set A and estimate the above bound of f on A . Let

$$\begin{aligned} Z_i &= \{ \nu(t) = Ri(\xi \cos 2\pi t + \eta \sin 2\pi t) | \xi, \eta \in \mathbb{R}^2, |\xi| = |\eta| = 1, \langle \xi, \eta \rangle = 0 \}, \\ A &= Z_1 \times \dots \times Z_N. \end{aligned}$$

Then for any $u = (u_1, \dots, u_N) \in A$, there are $\xi_i, \eta_i \in \mathbb{R}^2$ such that $|\xi_i| = |\eta_i| = 1$, $\xi_i \cdot \eta_i = 0$ and $u_i(t) = Ri(\xi_i \cos 2\pi t + \eta_i \sin 2\pi t)$. Hence $|u_i(t)|^2 = R^2 i^2$ and

$$\begin{aligned} |u_i(t) - u_j(t)|^2 &\geq |u_i(t)|^2 + |u_j(t)|^2 - 2|u_i(t)| \cdot |u_j(t)| \\ &= R^2(i^2 + j^2 - 2ij) = [R(i - j)]^2. \end{aligned}$$

Hence by (V2), we have

$$-V(u) \leq \frac{1}{R^\alpha} \left(\frac{a}{2} \sum_{i \neq j} \frac{m_i m_j}{|i - j|^\alpha} \right) = c_3 \cdot \frac{1}{R^\alpha}.$$

It is easily seen that

$$\begin{aligned} \dot{u}_i &= 2\pi R i (-\xi_i \sin 2\pi t + \eta_i \cos 2\pi t), \\ \|u\|_E^2 &= \sum_{i=1}^N m_i \int_0^1 |\dot{u}_i|^2 dt = R^2 \left(\sum_{i=1}^N 4\pi^2 i^2 m_i \right) = c_4 R^2. \end{aligned}$$

By the above estimates, we have

Lemma 2.7. *If we choose the R of the set A as $R^* = [(\alpha - 2)c_4/2h]^{1/\alpha}$, then*

$$\max_{u \in A} f(u) \leq c_5.$$

Proof.

$$\begin{aligned} f(u) &= \frac{1}{2} \|u\|_E^2 \cdot \int_0^1 (h - V(u)) dt \leq \frac{1}{2} c_4 R^2 \int_0^1 \left(h + c_3 \frac{1}{R^\alpha} \right) dt \\ &= \frac{1}{2} c_4 h R^2 + \frac{1}{2} c_3 c_4 R^{2-\alpha} = g(R). \end{aligned}$$

Then $R = R^* = [(\alpha - 2)c_4/2h]^{1/\alpha}$ satisfies $g'(R^*) = 0$, $g''(R^*) > 0$. So $g(R)$ has unique minimum value

$$g(R^*) = \frac{1}{2} c_4^{2/\alpha} \left(\frac{\alpha - 2}{2} \right)^{2/\alpha} h^{(\alpha-2)/\alpha} \left(c_4 + \frac{2}{\alpha - 2} c_3 \right) \equiv c_5 h^{(\alpha-2)/\alpha}.$$

In the following, we apply the equivariant Ljusternik-Schnirelmann theory to prove the existence of multiple geometrically distinct periodic solutions for (Ph).

We note that A is $S^1 \times O(2)$ invariant, in fact,

$$H(\tau, R) \cdot x(t) \equiv (Rx_1(t + \tau), \dots, Rx_N(t + \tau)) \in A$$

for any $x(t) = (x_1(t), \dots, x_N(t))$, $\tau \in S^1$ and $R \in O(2)$. We also note that the functional f is $S^1 \times O(2)$ invariant. Hence in order to obtain multiple distinct $S^1 \times O(2)$ invariant orbits of the system (Ph), we need to estimate the $S^1 \times O(2)$ equivariant category of the special set A . We note that each Z_i ($i = 1, \dots, N$) is diffeomorphic to $T_1 S^1$, which is the unit tangent bundle of S^1 . Hence $\text{cat}(Z_i) \geq 3$. Now $A = Z_1 \times \dots \times Z_N$ is diffeomorphic to $T_1 S^1 \otimes \dots \otimes T_1 S^1$ (N factors), and A is diffeomorphic to the set $Z_{(n_1, \dots, n_N)}$ of Coti Zelati ([4]), so by [4], we have

Lemma 2.8. $\text{cat}(A/S^1 \times O(2)) \geq 2(N - 1)2^{N-2}$.

Remark. Although the set A is diffeomorphic to the critical manifold $Z_{(n_1, \dots, n_N)}$ of Coti Zelati ([4]), the above bound of f on A is more easily estimated than on $Z_{(n_1, \dots, n_N)}$.

Lemma 2.9. *Let X be a Banach space, Λ an open subset of X and $f \in C^1(\Lambda, R)$. Let G be a compact Lie group, $T(G)$ a linear continuous representation with equivariant distance and M a C^{2-0} submanifold of Λ . Assume M and f both are*

invariant under $T(G)$, f satisfies the Palais-Smale condition on some closed subset N of M and the following boundary condition holds:

$$(B.C.) \quad f(u_n) \rightarrow +\infty, \quad \forall u_n \rightarrow u \in \partial\Lambda.$$

Let i be a $T(G)$ -invariant index. Let $c_m = \inf_{i(A) \geq m} \sup_{x \in A} f(x)$, $m = 1, 2, \dots$, where $A \subset N$, $A \in \Sigma = \{B \subset N \mid B \text{ is } T(G)\text{-invariant and closed in } N\}$. Then:

- (1) When $-\infty < c_m < +\infty$, c_m is a critical value of f .
- (2) If $-\infty < c = c_{m+1} = \dots = c_{m+k} < +\infty$, then $i(K_c) \geq k$, where $K_c = \{x \in N \mid f'(x) = 0, f(x) = c\}$.
- (3) $c_m \leq c_{m+1}$.

We note that, from the boundary condition (B.C.) and the Palais-Smale condition, one can deduce the complete property of the open set Λ , that is, for any $\{x_n\} \subset \Lambda$ such that $x_n \in f_c = \{x \in \Lambda \mid f(x) \leq c\}$ and $x_n \rightarrow x$, we have $x \in \Lambda$. The rest of the proof is standard (see Rabinowitz [10]).

Lemma 2.10. *If u is a critical point of f in Λ_0 which has minimal period $\frac{1}{l}$, $l \neq 1$, then*

- (i) $l \geq 3$,
- (ii) $v(t) = u(t/l)$ is also a critical point of f in Λ_0 and $f(v) = \frac{1}{l^2}f(u) < \frac{1}{9}f(u)$.

Proof. (i) If $l = 2$, then $u(t + \frac{1}{2}) = \bar{u}(t)$. By $u \in \Lambda_0$, we have $u(t + \frac{1}{2}) = -u(t)$. Hence $u(t) = (u_1(t), \dots, u_N(t)) \equiv 0$; this contradicts $u \in \Lambda_0$.

(ii)

$$\begin{aligned} \int_0^1 |\dot{v}(t)|^2 dt &= \frac{1}{l^2} \int_0^1 |\dot{u}(t)|^2 dt, \\ \int_0^1 (h - V(v)) dt &= \int_0^1 (h - V(u)) dt, \\ f(v) &= \frac{1}{l^2}f(u) \leq \frac{1}{9}f(u). \end{aligned}$$

Now Theorem 1.3 can be proved by Lemma 2.1 and Lemmas 2.4–2.10.

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