

ON A CONJECTURE OF DANZER AND GRÜNBAUM

MEIR KATCHALSKI AND DAVID NASHTIR

(Communicated by Jeffrey N. Kahn)

ABSTRACT. The main result of the paper is that if A is a family of homothetic triangles in the plane such that any 9 of them can be pierced by two points, then all members of A can be pierced by two points. This is best possible in more than one sense: (1) the number 9 cannot be replaced by 8; (2) no similar statement is true for homothetic copies (or even translates) of a symmetric convex hexagon.

1. INTRODUCTION AND STATEMENT OF RESULTS

A family of sets is Π^n , or n -pierceable, if there exists a set of n points, such that each member of the family contains at least one of the points. The family is Π_k^n if each subfamily of size k or less satisfies Π^n ; it is $-\Pi^n$ if it is not Π^n .

Helly's theorem asserts that Π_{d+1}^1 implies Π^1 for any family \mathcal{A} of compact convex sets in Euclidean d -space R^d . Consult [1] and [3] for Helly's theorem and its many relatives.

Danzer and Grünbaum studied in [2] intersection properties of boxes with sides parallel to the axis in R^d . They proved:

D. G. Theorem. *If $h = h(d, n)$ is the smallest integer such that Π_h^n implies Π^n for any family of boxes in R^d with sides parallel to the axis, then $h(d, 2) = 3d$ for odd d and $h(d, 2) = 3d - 1$ for even d .*

Here (section 2) a related result on families of homothetic triangles in the plane is proved.

Theorem 1. (i) *If \mathcal{A} is any family of homothets of a triangle in R^2 , then Π_9^2 implies Π^2 for \mathcal{A} .*

(ii) *There exists in R^2 a family of 9 translates of a triangle T satisfying Π_8^2 , but not Π^2 .*

Note that S is homothetic to B if there exists an $a \in R^d$ and an $\alpha > 0$ such that $S = a + \alpha B$. (Sometimes such an S is called a positive homothet of B .)

In the last section of [2], Danzer and Grünbaum raised two conjectures:

Conjecture 1. *If K is any convex polytope in R^d , then there exists an integer $k = k(K, d)$ such that Π_k^2 implies Π^2 for any family of translates of K in R^d .*

Received by the editors July 18, 1994.

1991 *Mathematics Subject Classification.* Primary 52A35.

The first author's research was supported by the Fund for Promotion of Research at the Technion (grant 100-806) and the Technion V. P. R. Fund (grant 100-934).

Conjecture 2. *If D is a set of m directions in the plane, then Π_{3m}^2 implies Π^2 for any family \mathcal{A} of convex polygons in the plane, all edges of which have directions belonging to D .*

In section 3 a counterexample to both conjectures is constructed: For any positive integer k a planar family of $2k + 3$ translates of a centrally symmetric hexagon satisfying Π_{2k+2}^2 but not Π^2 .

The last section describes related results.

2. PROOF OF THEOREM 1

The following definitions will be used.

Two halfplanes are *related* if one of them contains the other. Related halfplanes are ordered by inclusion so that the smaller one is contained in the bigger one.

A convex n -gon B is *related* to the convex m -gon A if each of the n -halfplanes whose intersection is equal to B , is related to one of the m -halfplanes whose intersection is equal to A . Note that this implies that $n \leq m$. The family \mathcal{B} is related to A if every member of \mathcal{B} is related to A .

Note that triangle B is related to triangle A if and only if B is homothetic to A .

A set Q pierces a set A if $A \cap Q \neq \emptyset$. The set Q pierces a family of sets \mathcal{A} if it pierces every $A \in \mathcal{A}$; x pierces \mathcal{A} if $\{x\}$ pierces \mathcal{A} .

By an m -gon we mean, unless stated otherwise, a convex m -gon.

Let X^+ denote a halfplane whose boundary is the line X .

Proof of Theorem 1(i). Let \mathcal{A} be a family of triangles related to a triangle T and satisfying Π_3^2 . Using compactness arguments, it is sufficient to prove part (i) for \mathcal{A} of finite cardinality.

Let $\mathcal{A} = \{A_i : i \in [n]\}$, $[n] = \{1, 2, 3, \dots, n\}$, where $A_i = H_i^+ \cap V_i^+ \cap D_i^+$ and $\{H_i^+; i \in [n]\}$, $\{D_i^+; i \in [n]\}$, $\{V_i^+; i \in [n]\}$ are three families of related halfplanes.

For $A = A_i$ let $D^+(A) = D_i^+$, $H^+(A) = H_i^+$, and $V^+(A) = V_i^+$.

For $\mathcal{C} \subseteq \mathcal{A}$ let $D^+(\mathcal{C}) = \min\{D^+(A); A \in \mathcal{C}\}$, $H^+(\mathcal{C}) = \min\{H^+(A); A \in \mathcal{C}\}$, and $V^+(\mathcal{C}) = \min\{V^+(A); A \in \mathcal{C}\}$.

Let $D^+ = D^+(\mathcal{A})$, $V^+ = V^+(\mathcal{A})$, and $H^+ = H^+(\mathcal{A})$ with $H^+ = H^+(T_3)$, $V^+ = V^+(T_2)$ and $D^+ = D^+(T_1)$.

Let $\mathcal{T} = \{T_1, T_2, T_3\}$.

Let $a_1 = H \cap V$, $a_2 = H \cap D$ and $a_3 = V \cap D$.

Assume that \mathcal{A} is $-\Pi^1$, for otherwise $\Pi^1 \Rightarrow \Pi^2$ (see Figure 1).

The proof relies on two simple observations:

Observation 1. For $\mathcal{C} \subseteq \mathcal{A}$ there exists a $\tilde{\mathcal{C}} \subseteq \mathcal{C}$ with $|\tilde{\mathcal{C}}| \leq 3$ and $\bigcap \tilde{\mathcal{C}} = \bigcap \mathcal{C}$. Furthermore, if $\mathcal{C} \cap \mathcal{T} \neq \emptyset$, then such a $\tilde{\mathcal{C}}$ can be chosen so that $\tilde{\mathcal{C}} \cap \mathcal{T} \neq \emptyset$.

Observation 2. If $\{i, j, k\} = \{1, 2, 3\}$ and x pierces \mathcal{B} with $\{T_i, T_j\} \subseteq \mathcal{B}$, then a_k pierces \mathcal{B} .

Observation 1 holds since $\bigcap \mathcal{C} = D^+(\mathcal{C}) \cap H^+(\mathcal{C}) \cap V^+(\mathcal{C})$ and if $D^+(\mathcal{C}) = D^+(F)$, $H^+(\mathcal{C}) = H^+(G)$, and $V^+(\mathcal{C}) = V^+(K)$, with $\tilde{\mathcal{C}} = \{F, G, K\} \subseteq \mathcal{C}$, then $\tilde{\mathcal{C}}$ is as desired.

Observation 2 holds since if w.l.o.g. x pierces \mathcal{B} with $\{T_1, T_2\} \subseteq \mathcal{B}$ and $A \in \mathcal{B}$, then $x \in A \cap T_1 \cap T_2$. Then $x \in D^+ \cap V^+$ and since $T_1 \cap T_2 \subset D^+ \cap V^+$, this implies $a_3 \in A$, for otherwise $A \cap D^+ \cap V^+ = \emptyset$, a contradiction.

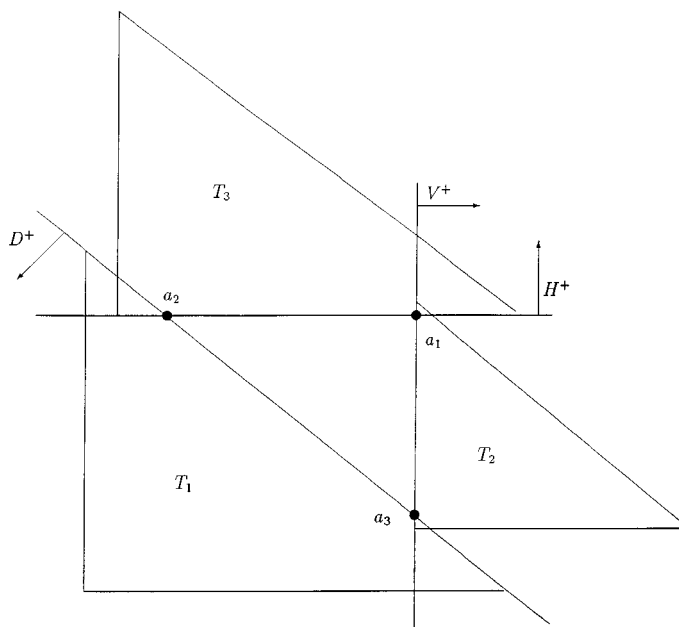


FIGURE 1

Let $\mathcal{A}_i = \{A \in \mathcal{A} : a_i \notin A\}$ for $i = 1, 2, 3$. Note that $T_i \in \mathcal{A}_i$ for $i = 1, 2, 3$. We claim that:

Claim. For $i = 1$ or 2 or 3 , \mathcal{A}_i is Π^1 .

The claim clearly completes the proof. *Otherwise* $\bigcap \mathcal{A}_i = \emptyset$ for $i = 1, 2, 3$ and by Observation 1, for $i = 1, 2, 3$ there exists a $\mathcal{C}_i \subseteq \mathcal{A}_i$, with $|\mathcal{C}_i| \leq 3$, $\bigcap \mathcal{C}_i = \emptyset$ and $\mathcal{C} \cap \mathcal{T} \neq \emptyset$.

Let

$$\mathcal{S} = \mathcal{T} \cup \mathcal{C}_1 \cup \mathcal{C}_2 \cup \mathcal{C}_3.$$

Then $|\mathcal{S}| \leq 3 + 2 + 2 + 2 = 9$. So $(\mathcal{A}$ is Π_9^2), \mathcal{T} is Π^2 and a 2 set $\{x, y\}$ pierces \mathcal{S} . Assume w.l.o.g. that x pierces $\{T_1, T_2\}$. By Observation 2 $\{a_3, y\}$ pierces \mathcal{S} and y pierces \mathcal{C}_3 , a contradiction, since $\bigcap \mathcal{C}_3 = \emptyset$. \square

Proof of Theorem 1(ii). A family $\mathcal{A} = \{A_i : 1 \leq i \leq 9\}$ of 9 translates of the triangle $T = \{(x, y) : x \geq 0, y \geq 0, x+y \leq 7\}$ satisfying Π_8^2 but not Π^2 is constructed as follows: for $1 \leq i \leq 9$, $A_i = a_i + T$, with

$$\begin{aligned} a_1 &= (0, -1), & a_2 &= (6, 0), & a_3 &= (-1, 6), \\ a_4 &= (0, 1), & a_5 &= (4, 1), & a_6 &= (0, 4), \\ a_7 &= (4, 0), & a_8 &= (1, 0), & a_9 &= (1, 4) \end{aligned}$$

(see Figure 2).

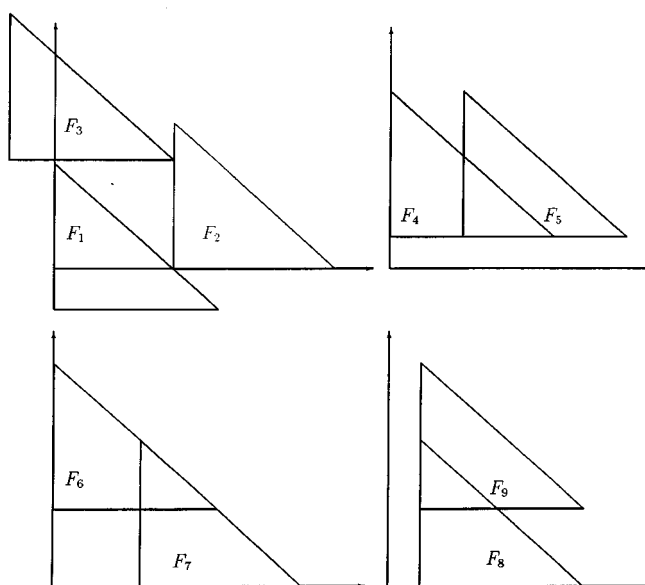


FIGURE 2

Let $\bar{\mathcal{A}}(i) = \mathcal{A} \setminus \{A_i\}$ for $1 \leq i \leq 9$. Then,

$$\begin{aligned} \{(6, 6), (4, 4)\} &\in \bar{\mathcal{A}}(1), & \{(4, 7), (6, 0)\} &\in \bar{\mathcal{A}}(4), & \{(1, 4), (6, 6)\} &\in \bar{\mathcal{A}}(7), \\ \{(0, 6), (4, 4)\} &\in \bar{\mathcal{A}}(2), & \{(1, 7), (6, 0)\} &\in \bar{\mathcal{A}}(5), & \{(7, 4), (0, 6)\} &\in \bar{\mathcal{A}}(8), \\ \{(6, 0), (4, 4)\} &\in \bar{\mathcal{A}}(3), & \{(4, 1), (6, 6)\} &\in \bar{\mathcal{A}}(6), & \{(7, 1), (0, 6)\} &\in \bar{\mathcal{A}}(9), \end{aligned}$$

so $\mathcal{A} \in \Pi_8^2$.

Since $A_1 \cap A_2 = \{(6, 0)\}$, $A_1 \cap A_3 = \{(0, 6)\}$ and $A_2 \cap A_3 = \{(6, 6)\}$, then a set \mathcal{B} of size 2 that pierces \mathcal{A} contains one of the points $(6, 0)$ or $(0, 6)$ or $(6, 6)$. But $(0, 6) \notin A_2 \cup A_8 \cup A_9$ and $A_2 \cap A_8 \cap A_9 = \emptyset$, so $(0, 6) \notin \mathcal{B}$. Also $(0, 6) \notin A_3 \cup A_4 \cup A_5$ and $A_3 \cap A_4 \cap A_5 = \emptyset$, so $(6, 0) \notin \mathcal{B}$. Finally, $(6, 6) \notin A_1 \cup A_6 \cup A_7$ and $A_1 \cap A_6 \cap A_7 = \emptyset$, so $(6, 6) \notin \mathcal{B}$. So there is no set \mathcal{B} of size 2 that pierces \mathcal{A} and \mathcal{A} is not Π^2 . \square

3. CONSTRUCTIONS

Danzer and Grünbaum's theorem asserts that Π_5^2 implies Π^2 for planar families related to a square (the members of such a family are necessarily rectangles with sides parallel to the sides of the square). It is natural to attempt to generalize and prove a similar theorem for planar families of convex sets related to a hexagon. In view of the following construction these attempts cannot succeed.

The construction consists of a family of translates of a centrally symmetric hexagon.

Construction 1. Consider a 2×2 square with sides parallel to the axis. Let c and f be opposite vertices with f above and to the right of c . Let a, b, d and e be the midpoints of the four sides of the square with a above d and e to the right of b .

Let H denote the centrally symmetric hexagon with vertices a, b, c, d, e and f .

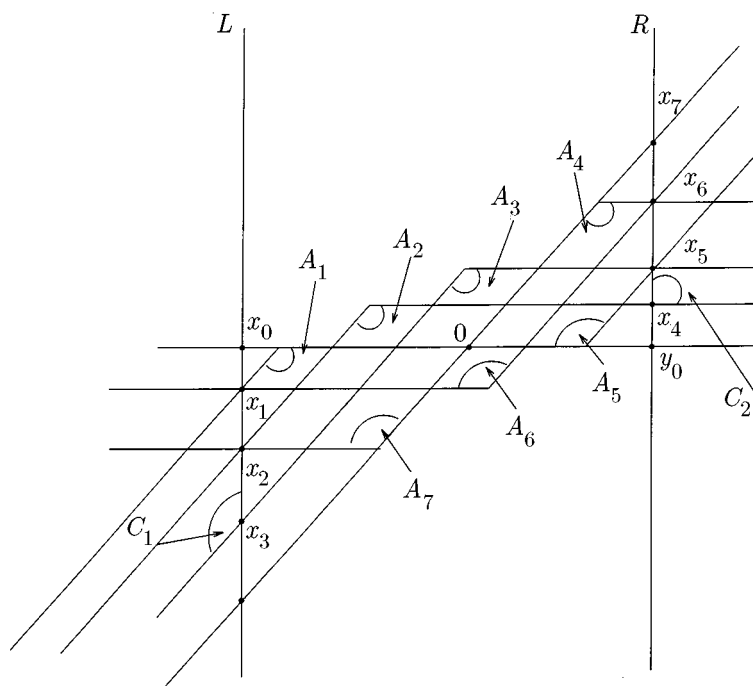


FIGURE 3

Let L and R be two vertical lines with distance between them equal to 1 and with L to the left of R . Let x_0 and y_0 lie on a horizontal line with x_0 on L and y_0 on R .

Let k be a positive integer and let z_0 on L and x_{2k+1} on R lie on a line of slope 1 (and therefore parallel to segment ab of the hexagon H) that contains the midpoint 0 of segment x_0y_0 .

Let x_1, \dots, x_k lie on L between x_0 and z_0 with x_{i+1} below x_i for $1 \leq i \leq k - 1$.

Let $x_{k+1}, x_{k+2}, \dots, x_{2k}$ lie on R between y_0 and x_{2k+1} with x_{k+i+1} above x_{k+i} for $1 \leq i \leq k$ (see Figure 3 where $k = 3$). Define translates $A_1, \dots, A_{2k+1}, C_1$ and C_2 of H as follows:

The ab -side (corresponding to side ab of H) of A_i contains x_i for $1 \leq i \leq k$.

The ab -side of A_{k+1} contains 0.

The af -side of A_1 contains 0.

The af -side of A_i contains x_{i+k-1} for $2 \leq i \leq k + 1$.

The ed -side of A_i contains x_i for $k + 2 \leq i \leq 2k + 1$.

The cd -side of A_{k+i+1} contains x_{i-1} for $1 \leq i \leq k$.

Finally:

The e -vertex of C_1 is x_k .

The c -vertex of C_2 is x_{k+1} (see Figure 3).

However, in the figure, suitable angles of the hexagons and not the complete hexagons are sketched (the a -angle of A_i for $1 \leq i \leq k + 1$, the d -angle of A_i for $k + 2 \leq i \leq 2k + 1$, the c -angle of C_2 and the e -angle of C_1).

Let $\mathcal{A} = \{A_1, A_2, \dots, A_{2k+1}, C_1, C_2\}$. It remains to show that

Theorem 2. *The family \mathcal{A} of Construction 2 is Π_{2k+2}^2 but not Π^2 .*

Proof. For $X \in \mathcal{A}$ let $\overline{\mathcal{A}}(X)$ denote $\mathcal{A} \setminus \{X\}$. All the A_i 's contain 0 so $\{0, x_{k+1}\}$ pierces $\overline{\mathcal{A}}(C_1)$ and $\{0, x_k\}$ pierces $\overline{\mathcal{A}}(C_2)$. Since A_i contains $\{x_i, x_{i+1}, \dots, x_{i+k-1}\}$ (with addition modulo $2k+1$), it follows that $\{x_{i+k}, x_{i+2k}\}$ pierces $\mathcal{A} \setminus \{A_i, C_1, C_2\}$. But $\{x_{i+k}, x_{i+2k}\}$ pierces $\overline{\mathcal{A}}(A_i)$ for $2 \leq i \leq 2k+1$. However, $\{x_0, x_{k+1}\}$ pierces $\overline{\mathcal{A}}(A_1)$. So

$$\mathcal{A} \text{ is } \Pi_{2k+2}^2.$$

Assume that \mathcal{A} is π^2 and that $\{u, v\}$ pierces \mathcal{A} . Since C_1 and C_2 are disjoint, assume that $u \in C_1$ and $v \in C_2$. It is easy to check that if we project u and v horizontally to L and R respectively, then the set consisting of the two projections pierces \mathcal{A} . It is also easy to check that a point of $L \cap C_1$ as well as a point of $R \cap C_2$ pierces at most k members of $\{A_1, \dots, A_{2k+1}\}$. Consequently, $\{u, v\}$ pierces at most $2k+2$ members of \mathcal{A} , a contradiction, implying that \mathcal{A} is not Π^2 . \square

4. RELATED RESULTS

Related results may be found in [4], the M.Sc. Thesis of David Nashtir (written under the supervision of Meir Katchalski).

(1) A generalization of Theorem 1.

Theorem. *Let P be a convex pentagon with two pairs of parallel sides. If \mathcal{A} is any planar family of convex sets related to P and if \mathcal{A} is Π_5^2 , then \mathcal{A} is Π^2 .*

The proof is very technical.

(2) Constructions.

Construction 2. A convex quadrilateral Q and for every positive integer k a family \mathcal{A} of $2k+3$ translates of Q satisfying Π_{2k+2}^2 but not Π^2 .

The construction is similar to Construction 1.

Construction 3. For every positive integer k a planar family of $3n+1$ triangles related to a triangle T and satisfying Π_{3n}^3 but not Π^3 .

Additional results will appear in Nashtir's Ph.D. thesis [5] (written under the supervision of Katchalski). It is proved there that there is no positive integer k such that Π_k^2 implies Π^2 for every family of translates of a fixed nondegenerate simplex in R^3 .

Other implications of Π_k^2 for families of convex sets are also studied there.

REFERENCES

1. L. Danzer, B. Grünbaum and V. Klee, *Helly's theorem and its relatives*, Proc. Symposia in Pure Math., vol. VII (Convexity) (1963), 101–180. MR **28**:524
2. L. Danzer and B. Grünbaum, *Intersection properties of boxes in R^d* , Combinatorica **2** (1982), 237–246. MR **84g**:52014
3. J. Eckhoff, *Helly Radon and Carathéodory Type Theorems*, Chapter 2.1 in P. M. Gruber and J. M. Wills (eds.), Handbook of Convex Geometry, North-Holland (1993), 389–448. MR **94k**:52010
4. D. Nashtir, *On a conjecture of Danzer and Grünbaum*, Ms.C. Thesis, Technion, Haifa (1990), (Hebrew)
5. ———, *Helly type problems*, Ph.D. Thesis, in preparation.

DEPARTMENT OF MATHEMATICS, TECHNION–ISRAEL INSTITUTE OF TECHNOLOGY, HAIFA 32000, ISRAEL

E-mail address: meirk@tx.technion.ac.il