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ON MAXIMAL FUNCTIONS IN ORLICZ SPACES

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ABSTRACT. Let $\Phi(t)$ and $\Psi(t)$ be the functions having the representations $\Phi(t) = \int_0^t a(s)ds$ and $\Psi(t) = \int_0^t b(s)ds$, where $a(s)$ is a positive continuous function such that $\int_1^\infty \frac{a(s)}{s}ds = +\infty$ and $b(s)$ is quasi-increasing. Then the maximal function Mf is a function in Orlicz space L^Φ for all $f \in L^\Psi$ if and only if there exists a positive constant c_1 such that $\int_1^s \frac{a(t)}{t}dt \le c_1 b(c_1 s)$ for all $s \ge 1$.

1. INTRODUCTION

Let T be the group of real numbers modulo 2π, and let $f(x)$ be a real valued integrable function defined on T with period 2π. The classical Hardy-Littlewood maximal function $Mf(x)$ is defined by

$$Mf(x) := \sup_{x\in I} \frac{1}{|I|} \int_I |f(y)|dy, \tag{1.1}$$

where the supremum is taken over all open intervals $I \subset T$ with $x \in I$.

The aim of this paper is to give a necessary and sufficient condition such that the Hardy-Littlewood maximal function $Mf(x)$ defined by (1.1) is a function in Orlicz space L^Φ, whenever $f(x)$ is an arbitrary function in Orlicz space L^Ψ. Orlicz space L^Ψ is defined as follows.

Definition 1.1. Let $\Psi(t)$ be a nondecreasing continuous function such that $\lim_{t\to\infty} \Psi(t) = +\infty$. Put

$$L^\Psi := \left\{ f : \int_0^{2\pi} \Psi(\varepsilon|f(x)|)dx < +\infty \quad \text{for some} \quad \varepsilon > 0 \right\}. \tag{1.2}$$

Then the space L^Ψ is called an Orlicz space (see Kita and Yoneda [2], Rao and Ren [4] and Zygmund [6]).

We note that if $L^\Psi(t) = t^p$ for $t \ge 0$ and $p > 1$, then L^Ψ is a usual Lebesgue space.

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2. Main theorems

As is well known, the sublinear operator M is of type (p,p) $(1 < p \leq \infty)$ and weak type (1,1) (see Torchinsky [5] and Zygmund [6]). For the maximal functions in the class $\Phi(L) := \{f : \int_{\mathbf{R}^n} \Phi(|f(x)|)dx < +\infty\}$, the detailed results can be found in the monograph of Kokilashvili and Krbec [3]. In this paper the maximal function of a function in Orlicz function space L^Ψ defined by (1.2) will be considered.

A function $b(s)$ defined on $[0,\infty)$ is called quasi-increasing if there exists a positive constant c_0 such that $b(s_1) \leq c_0 b(c_0 s_2)$ for all $0 < s_1 < s_2$.

Let $a(s)$ and $b(s)$ be positive continuous functions defined on $[0,\infty)$ satisfying the following properties:

$$\int_1^\infty \frac{a(s)}{s} ds = +\infty \ ; \tag{2.1}$$

$$b(s) \text{ is quasi-increasing} \quad \text{and} \quad \lim_{s\to\infty} b(s) = +\infty. \tag{2.2}$$

Put

$$\Phi(t) := \int_0^t a(s)ds \quad \text{and} \quad \Psi := \int_0^t b(s)ds \quad \text{for} \quad t \geq 0. \tag{2.3}$$

We have the following result which is also a generalization of the result in [5], p.103.

Theorem 2.1. *Let $a(s)$, $b(s)$, $\Phi(t)$ and $\Psi(t)$ be the functions satisfying the above properties* (2.1) – (2.3). *Then the following statements are equivalent:*

(i) *there exists a positive constant c_1 such that*

$$\int_1^s \frac{a(t)}{t} dt \leq c_1 b(c_1 s) \quad for\ all \quad s \geq 1; \tag{2.4}$$

(ii) *there exists a positive constant c_2 such that*

$$\int_0^{2\pi} \Phi(Mf(x))dx \leq c_2 + c_2 \int_0^{2\pi} \Psi(c_2|f(x)|)dx \quad for\ all \quad f \in L^1(T). \tag{2.5}$$

As a corollary of Theorem 2.1 we have the following result.

Theorem 2.2. *Let $a(s), b(s), \Phi(t)$ and $\Psi(t)$ be the functions satisfying* (2.1)–(2.3). *Then the maximal function Mf is a function in Orlicz space L^Φ for all $f \in L^\Psi$ if and only if the functions $a(s)$ and $b(s)$ satisfy the inequality* (2.4).

We note that in Theorems 2.1 and 2.2 the functions $\Phi(t)$ and $\Psi(t)$ do not necessarily satisfy the Δ_2–condition, that is, there exist positive constants c and t_0 such that $\Phi(2t) \leq c\Phi(t)$ for all $t \geq t_0$.

Let us point out two particular consequences of the above theorems.

Corollary 2.3. *Let $a(s)$ be a positive continuous function defined on $[0,\infty)$ such that $\int_1^\infty \frac{a(s)}{s} ds = +\infty$. Put $b(s) := \int_1^s \frac{a(t)}{t} dt + 1$ for $s \geq 1$, $b(s) = 1$ for $0 \leq s < 1$, $\Phi(t) := \int_0^t a(s)ds$ and $\Psi(t) := \int_0^t b(s)ds$. Then for any Orlicz space $L^{\Psi_1} \not\subseteq L^\Psi$, there exists a function $f_0 \in L^{\Psi_1}$ such that $f_0 \notin L^\Psi$ and $Mf_0 \notin L^\Phi$.*

Corollary 2.4. *Let $a(s)$ be a positive quasi-increasing continuous function defined on $[0,\infty)$. Put $\Phi(t) := \int_0^t a(s)ds$. Then the following statements are equivalent:*

(i) *there exists a positive constant* c_1 *such that*

$$\int_1^s \frac{a(t)}{t} dt \le c_1 a(c_1 s) \quad \text{for all} \quad s \ge 1. \tag{2.6}$$

(ii) *there exists a positive constant* c_2 *such that*

$$\int_0^{2\pi} \Phi(Mf(x))dx \le c_2 + c_2 \int_0^{2\pi} \Phi(c_2|f(x)|)dx \quad \text{for all} \quad f \in L^1. \tag{2.7}$$

3. Proof of the theorems

First we prove Theorem 2.1. Let us prove (i) $\Rightarrow$ (ii). Let $\chi_{\{Mf>1\}}(x)$ be a characteristic function on the set $\{x \in T : Mf(x) > 1\}$ and put $F(x) := \Phi(Mf(x))\chi_{\{Mf>1\}}(x)$. Then it follows that

$$\begin{aligned} I := \int_{Mf>1} \Phi(Mf(x))dx &= \int_0^{2\pi} F(x)dx = \int_0^{\infty} |\{F > \lambda\}| d\lambda \\ &= \int_0^{\Phi(1)} |\{F > \lambda\}| d\lambda + \int_{\Phi(1)}^{\infty} |\{F > \lambda\}| d\lambda \\ &\le 2\pi\Phi(1) + \int_{\Phi(1)}^{\infty} |\{F > \lambda\}| d\lambda \ . \end{aligned}$$

From (2.1) and (2.3), $\Phi(t)$ is strictly increasing. Therefore we get

$$I \le 2\pi\Phi(1) + \int_{\Phi(1)}^{\infty} |\{Mf > \Phi^{-1}(\lambda)\}| d\lambda \ . \tag{3.1}$$

Put $t = \Phi^{-1}(\lambda)$. From (2.1) and (2.3) $\lim_{t\to\infty} \Phi(t) = +\infty$ holds. Therefore we have

$$\int_{\Phi(1)}^{\infty} |\{F > \lambda\}| d\lambda = \int_1^{\infty} |\{Mf > t\}| a(t) dt \ . \tag{3.2}$$

Since the sublinear operator M is simultaneously of weak-type (1,1) and of type (∞, ∞), it follows by the well known result (see Torchinsky [5], p.92) that there exist positive constants c_3 and c_4 such that

$$|\{Mf > t\}| \le \frac{c_3}{t} \int_{t/c_4}^{\infty} |\{|f| > s\}| ds \quad \text{for all} \quad t > 0 \ . \tag{3.3}$$

Therefore it follows from (3.1), (3.2) and (3.3) that

$$\begin{aligned} I &\le 2\pi\Phi(1) + c_3 \int_1^{\infty} \frac{a(t)}{t} \left(\int_{t/c_4}^{\infty} |\{|f| > s\}| ds \right) dt \\ &= 2\pi\Phi(1) + c_3 \int_{1/c_4}^{\infty} \left(|\{|f| > s\}| \int_1^{c_4 s} \frac{a(t)}{t} dt \right) ds \ . \end{aligned}$$

By the assumption (i), we have

$$\begin{aligned} I &\le 2\pi\Phi(1) + c_1 c_3 \int_{1/c_4}^{\infty} |\{|f| > s\}| b(c_1 c_4 s) ds \\ &\le 2\pi\Phi(1) + (c_3/c_4) \int_0^{2\pi} \Psi(c_1 c_4 |f(x)|) dx \ . \end{aligned}$$

We conclude that (2.5) holds.

Next we prove (ii) $\Rightarrow$ (i). Let (ii) hold and we assume that (i) does not hold. Then there exists a sequence of numbers $\{s_k : k \geq 1\}$ such that $s_k \geq 1$ for $k \geq 1$ and

$$\int_1^{s_k} \frac{a(t)}{t} dt > 2^k b(k2^k s_k) \quad \text{for} \quad k \geq 1 \ . \tag{3.4}$$

We choose a collection of disjoint open intervals $\{I_k : k \geq k_0\}$, $I_k \subseteq T$, such that

$$|I_k| := \frac{1}{2^k \Psi(2^k s_k)} \quad \text{for} \quad k \geq k_0 \quad \text{and} \quad \sum_{k=k_0}^{\infty} |I_k| < 2\pi \ . \tag{3.5}$$

Put

$$f(x) := \frac{\varepsilon_0}{c_2} \sum_{k=k_0}^{\infty} 2^k s_k \chi_{I_k}(x) \ , \qquad 0 < \varepsilon_0 < 1 \ , \tag{3.6}$$

where χ_{I_k} is a characteristic function on I_k and a number ε_0 will be defined later. Then it follows from (3.5) and (3.6) that

$$\begin{aligned} \int_0^{2\pi} \Psi(c_2|f(x)|)dx &= \sum_{k=k_0}^{\infty} \int_{I_k} \Psi(c_2|f(x)|)dx \\ &= \sum_{k=k_0}^{\infty} \Psi(\varepsilon_0 2^k s_k)|I_k| \leq \sum_{k=k_0}^{\infty} \Psi(2^k s_k) \cdot \frac{1}{2^k \Psi(2^k s_k)} \\ &= \sum_{k=k_0}^{\infty} 2^{-k} < +\infty \ . \end{aligned}$$

Therefore we get $f \in L^\Psi$. Since $b(s)$ is quasi-increasing, we have

$$\Psi(t) \geq \int_{t/2}^{t} b(s) \geq \frac{1}{c_0} \int_{t/2}^{t} b(\frac{t}{2c_0})ds = \frac{t}{2c_0} b(\frac{t}{2c_0}) \ .$$

Therefore $L^\Psi \subseteq L^1$ holds and we can choose ε_0 such that

$$|f|_T := \frac{1}{2\pi} \int_0^{2\pi} |f(x)|dx < 1 \ . \tag{3.7}$$

On the other hand, for each $0 < \varepsilon < 1$, $\int_0^{2\pi} \Phi(\varepsilon Mf(x))dx = +\infty$ holds. Indeed, put $g = \varepsilon f$. It is a well known fact (see Torchinsky [5], p.93) that there exists a positive constant c_5 such that

$$|\{Mg > \lambda\}| \geq \frac{c_5}{\lambda} \int_{|g|>\lambda} |g(x)|dx \quad \text{for all} \quad \lambda > |g|_T \ . \tag{3.8}$$

Therefore it follows from (3.7) and (3.8) that

$$\begin{aligned}\int_0^{2\pi} \Phi(\varepsilon Mf(x))dx &= \int_0^{2\pi} \Phi(Mg(x))dx = \int_0^{\infty} |\{Mg > \lambda\}|\Phi'(\lambda)d\lambda \\ &\geq \int_{|g|_T}^{\infty} |\{Mg > \lambda\}|\Phi'(\lambda)d\lambda \geq \int_1^{\infty} |\{Mg > \lambda\}|\Phi'(\lambda)d\lambda \\ &\geq c_5 \int_1^{\infty} \left(\int_{|g|>\lambda} |g(x)|dx\right) \frac{\Phi'(\lambda)}{\lambda} d\lambda \\ &= c_5 \int_1^{\infty} \left(\int_0^{2\pi} |g(x)|\chi_{\{|g|>\lambda\}}(x)dx\right) \frac{\Phi'(\lambda)}{\lambda} d\lambda \\ &= c_5 \int_0^{2\pi} |g(x)| \left(\int_1^{\infty} \chi_{\{|g|>\lambda\}}(x) \frac{\Phi'(\lambda)}{\lambda} d\lambda\right) dx \\ &= c_5 \int_{|g|>1} |g(x)| \left(\int_1^{|g(x)|} \frac{a(\lambda)}{\lambda} d\lambda\right) dx\ .\end{aligned}$$

Since $g(x) = \varepsilon f(x)$ and $f(x)$ is constant on I_k, we get

$$g(x) = \frac{\varepsilon\varepsilon_0}{c_2} 2^k s_k \geq \frac{\varepsilon\varepsilon_0 2^k}{c_2} \ \uparrow\ +\infty \quad \text{as} \quad k \uparrow +\infty\ .$$

Thus, there exists a positive integer $k_1 > k_0$ such that

$$\frac{\varepsilon\varepsilon_0 2^k}{c_2} > 1 \qquad \text{for all} \qquad k \geq k_1\ . \tag{3.9}$$

From (3.4), (3.5) and (3.9), it follows that

$$\begin{aligned}\int_0^{2\pi} \Phi(\varepsilon Mf(x))dx &\geq c_5 \sum_{k=k_1}^{\infty} \int_{I_k} |g(x)| \left(\int_1^{|g(x)|} \frac{a(\lambda)}{\lambda} d\lambda\right) dx \\ &= \frac{c_5\varepsilon\varepsilon_0}{c_2} \sum_{k=k_1}^{\infty} 2^k s_k \left(\int_1^{\frac{\varepsilon\varepsilon_0 2^k s_k}{c_2}} \frac{a(\lambda)}{\lambda} d\lambda\right) |I_k| \\ &\geq \frac{c_5\varepsilon\varepsilon_0}{c_2} \sum_{k=k_1}^{\infty} 2^k s_k \left(\int_1^{s_k} \frac{a(\lambda)}{\lambda} d\lambda\right) \cdot \frac{1}{2^k \Psi(2^k s_k)} \\ &\geq \frac{c_5\varepsilon\varepsilon_0}{c_2} \sum_{k=k_1}^{\infty} 2^k s_k \cdot 2^k s_k b(k 2^k s_k) \cdot \frac{1}{2^k \Psi(2^k s_k)}\ .\end{aligned}$$

Since $b(s)$ is quasi-increasing, it follows that

$$\Psi(2^k s_k) = \int_0^{2^k s_k} b(s)ds \leq \int_0^{2^k s_k} c_0 b(c_0 2^k s_k)ds = c_0 2^k s_k b(c_0 2^k s_k)\ .$$

Therefore we have

$$\begin{aligned}\int_0^{2\pi} \Phi(\varepsilon Mf(x))dx &\geq \frac{c_5\varepsilon\varepsilon_0}{c_2} \sum_{k=k_1}^{\infty} \frac{2^k s_k \cdot 2^k b(k 2^k s_k)}{2^k \cdot c_0 2^k s_k \cdot b(c_0 2^k s_k)} \\ &= \frac{c_5\varepsilon\varepsilon_0}{c_2 c_0} \sum_{k=k_1}^{\infty} \frac{b(k 2^k s_k)}{b(c_0 2^k s_k)}\ .\end{aligned}$$

If we choose a positive integer $k_2 > k_1$ such that $k_2 \geq c_0^2$, then $b(c_0 2^k s_k) \leq c_0 b(c_0 \cdot \frac{k}{c_0} 2^k s_k)$ for all $k \geq k_2$. Therefore we have

$$\int_0^{2\pi} \Phi(\varepsilon M f(x))dx \geq \frac{c_5 \varepsilon \varepsilon_0}{c_2 c_0^2} \sum_{k=k_2}^{\infty} 1 \; = \; +\infty \; .$$

We arrive at a contradiction and Theorem 2.1 is proved.

Last we prove Theorem 2.2. Let us assume that the inequality (2.4) holds. Let f be an arbitrary function in Orlicz space L^Ψ. Then there exists a positive number ε_1 such that $\int_0^{2\pi} \Psi(\varepsilon_1 |f(x)|)dx < +\infty$. By Theorem 2.1, we have the following inequality:

$$\int_0^{2\pi} \Phi(\frac{\varepsilon_1}{c_2} M f(x))dx \leq c_2 + c_2 \int_0^{2\pi} \Psi(\varepsilon_1 |f(x)|)dx < +\infty \; .$$

Thus $Mf \in L^\Phi$ holds.

Conversely, let us assume that (2.4) does not hold. Then, by the same manner as the proof of Theorem 2.1, it follows that there exists a function $f \in L^\Psi$ such that $Mf \notin L^\Phi$. This is a contradiction and the proof is complete.

Remark. Let $a(s)$ be a positive quasi-increasing continuous function defined on $[0, \infty)$. Put $\Phi(t) = \int_1^t a(s)ds$ for $t \geq 1$ and $\Phi(t) = 0$ for $0 \leq t < 1$. It is easy to see from the result given in [3] (see p.6, Theorem 1.2.1) that the following statements are equivalent:

(i) there exists a positive constant c_1 such that

$$\int_1^t \frac{a(s)}{s} ds \leq c_1 a(c_1 t) \quad \text{for all} \quad t \geq 1;$$

(ii) there exists a positive constant c_2 such that

$$\int_1^t \frac{\Phi(s)}{s^2} ds \leq \frac{c_2 \Phi(c_2 t)}{t} \quad \text{for all} \quad t \geq 1;$$

(iii) there exists a constant $\ell > 1$ such that

$$\Phi(t) \leq \frac{1}{2\ell} \Phi(\ell t) \quad \text{for all } t \geq 1;$$

(iv) the function Φ^α is quasi convex on $[0, \infty)$ for some $0 < \alpha < 1$, that is, there exist a convex function ω and a constant $c > 0$ such that $\omega(t) \leq \Phi(t) \leq c\omega(ct)$ for all $t \geq 0$.

We note that if $a(t)$ is an increasing function on $[0, \infty)$ satisfying the inequality in the statement of (i), then there exist positive constants α, t_0 and c_0 such that $a(t) \geq c_0 t^\alpha$ for all $t \geq t_0$ (see Rao and Ren [4], p.26, Corollary 5).

4. Examples

In this section, some examples of functions $\Phi(t), \Psi(t), a(t)$ and $b(t)$ will be given. Let $\varphi_1(t)$ and $\varphi_2(t)$ be the functions defined on $[0, \infty)$. We write $\varphi_1(t) \sim \varphi_2(t)$, if there exist positive constants c_1, c_2 and t_0 such that $c_1 \varphi_1(t) \leq \varphi_2(t) \leq c_2 \varphi_1(t)$ for all $t \geq t_0$.

Example 1. *Let* $1 < p < +\infty$.

$$\begin{cases} \Phi(t) = \frac{1}{p}t^p, & a(t) = t^{p-1} \quad \textit{for} \quad t \geq 0\,; \\ \Psi(t) = \frac{1}{p}t^p, & b(t) = t^{p-1} \quad \textit{for} \quad t \geq 0\,. \end{cases}$$

Example 2. *Let* $0 < \alpha \leq 1$.

$$\begin{cases} \Phi(t) \sim \frac{t}{(\log t)^{1-\alpha}}, & a(t) \sim \frac{1}{(\log t)^{1-\alpha}}; \\ \Psi(t) \sim t(\log t)^{\alpha}, & b(t) \sim (\log t)^{\alpha}. \end{cases}$$

Example 3.

$$\begin{cases} \Phi(t) \sim \dfrac{t}{\log t}, & a(t) \sim \dfrac{1}{\log t} \\ \Psi(t) \sim t(\log\log t), & b(t) \sim \log\log t. \end{cases}$$

Example 4. *Put* $\log^+ t = \log t$ *for* $t \geq 1$ *and* $\log^+ t = 0$ *for* $0 \leq t < 1$. *And put* $L_1(t) = \log^+ t$, $L_n(t) = \log^+ L_{n-1}(t)$ *for* $n \geq 2$.

$$\begin{cases} \Phi(t) \sim \dfrac{t}{L_1(t)L_2(t)\cdots L_{n-1}(t)}, & a(t) \sim \dfrac{1}{L_1(t)L_2(t)\cdots L_{n-1}(t)} \\ \Psi(t) \sim tL_n(t), & b(t) \sim L_n(t). \end{cases}$$

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