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A NEW PROOF OF MASSER'S VANISHING THEOREM

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Abstract. We give a simple proof of Masser's vanishing theorem, which is important in investigating the algebraic independence of the values of Mahler functions.

1. Introduction

Masser's vanishing theorem [4] is a complete solution to Problem 1 of Mahler's article [3], and is important in investigating the algebraic independence of values of Mahler functions. Masser's proof is divided into two parts. The first part is reproved in Nishioka and Nishioka [5] by using p-adic analysis. Their method derives more consequences (Propositions 1 and 2 below), which offer a rather straightforward proof of the vanishing theorem.

Let Ω be a matrix of n rows and n columns with entries ω_{ij} $(1 \leq i, j \leq n)$ being non-negative integers. For a vector $z = (z_1, \ldots, z_n)$ define Ωz as the vector $(\prod_{i=1}^n z_i^{\omega_{1i}}, \ldots, \prod_{i=1}^n z_i^{\omega_{ni}})$. We assume that the matrix Ω is non-singular and none of its eigenvalues is a root of unity. Let α be an algebraic point of $\mathbb{C}^n$ whose components $\alpha_1, \ldots, \alpha_n$ are non-zero algebraic numbers such that $\Omega^k \alpha \to (0, \ldots, 0)$ as $k \to \infty$.

Theorem (Masser [4])**.** *The following two conditions are equivalent:*

(i) *There exist a non-zero power series $f(z) \in \mathbb{C}[[z_1, \ldots, z_n]]$ and an integer k_0 such that $f(\Omega^k \alpha) = 0$ for all $k \geq k_0$.*

(ii) *There exist a non-zero difference of monomials $D(z)$ and positive integers a and m such that $D(\Omega^k \alpha) = 0$ for all $r = m + ak$ $(k \geq 0)$.*

2. Proof of the Theorem

Proposition 1. *Let C be an algebraically closed field of characteristic 0. Let $P(z)$ be a non-zero polynomial in $C[z_1, \ldots, z_n]$, and x a vector in C^n with non-zero components $x_1, \ldots, x_n$. We put $P(z) = \sum_{I=(i_1,\ldots,i_n)\in\Lambda} c_I z^I$ $(c_I \in C^\times,\ I \in \Lambda)$. If $P(\Omega^k x) = 0$ for infinitely many k, then there exist distinct elements $I, J \in \Lambda$ and positive integers a and m such that*

$$x^{(I-J)\Omega^m(\Omega^{ak}-E)} = 1, \qquad P(\Omega^{m+ak}x) = 0$$

for any $k \geq 0$, where E is the identity matrix.

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Proof. This is included in the proof of [5, Theorem 3] by setting $DM^2 = a$ therein.

Proposition 2. *Let C be an algebraically closed field of characteristic 0. Let $P(z)$ be the same as in Proposition 1, and S an infinite subset of the positive integers $\mathbb{N}$. Then there exist polynomials $B_1(z), \dots, B_s(z) \in C[z_1, \dots, z_n]$, $b_1, \dots, b_s \in S$, $u \in \mathbb{N}$ and monomials $U_1(z), \dots, U_t(z)$, $V_1(z), \dots, V_t(z)$ such that:*

(i) *For any j $(1 \le j \le t)$ there exist distinct $I, J \in \Lambda$ and $a, m \in \mathbb{N}$ such that*

$$U_j(z)/V_j(z) = z^{(I-J)\Omega^m(\Omega^a - E)}.$$

(ii) $\sum_{i=1}^{s} B_i(z)P(\Omega^{b_i} z) = (z_1 \cdots z_n)^u \prod_{j=1}^{t}(U_j(z) - V_j(z))$.

Proof. Let A be the affine algebraic set defined by $\{P(\Omega^k z)\}_{k \in S}$, and $W_1, \dots, W_r$ the irreducible components of A. Let $(x_1, \dots, x_n)$ be a generic point of W_j over C with components in some field extension of C. If $x_i = 0$ for a certain i, then W_j is included in the algebraic set defined by $z_i = 0$. If $x_i \ne 0$ for every i, then by Proposition 1 there exist distinct $I, J \in \Lambda$ and $a, m \in \mathbb{N}$ such that

$$x^{(I-J)\Omega^m(\Omega^a - E)} = 1.$$

There exist monomials U_j, V_j such that

$$U_j(z)/V_j(z) = z^{(I-J)\Omega^m(\Omega^a - E)}.$$

Since $U_j(x) - V_j(x) = 0$, W_j is included in the algebraic set defined by $U_j(z) - V_j(z) = 0$. Therefore we may assume that A is included in the algebraic set defined by

$$(z_1 \cdots z_n) \prod_{j=1}^{r'} (U_j(z) - V_j(z)),$$

where r' is the number of W's which are not included in the algebraic set defined by $z_i = 0$ for some i. By the Hilbert Nullstellensatz, we obtain the proposition.

For a subset R of $\mathbb{N}$, we define the density $\delta(R)$ of R by

$$\delta(R) = \limsup_{x \to \infty} x^{-1} |\{k \in R | 1 \le k \le x\}|.$$

Lemma (Masser [4], Lemma 3)**.** *Let R be a subset of $\mathbb{N}$ of positive density. Then there exists a strictly increasing sequence $d_1, d_2, \dots$ of non-negative integers with the following property. For each $s \ge 1$ let R_s be the set of non-negative integers r such that $r + d_1, \dots, r + d_s$ are all in R. Then R_s has positive density for all $s \ge 1$.*

Proof of the Theorem. Replacing Ω by a suitable positive power of itself, we may assume the multiplicative group generated by the eigenvalues of Ω is torsion free. Let $f(z)$ be as in (i) and put $f(z) = \sum_{I=(i_1,\dots,i_n)} c_I z^I$ $(c_I \in \mathbb{C})$. Then $|c_I| \le c_1 \cdot c_1 |I|$, where $|I| = i_1 + \cdots + i_n$. In what follows $c_1, c_2, \dots$ denote positive constants independent of k. Let $S = \{I = (i_1, \dots, i_n) | c_I \ne 0\}$. By Lemma 3 in Kubota [2], S has a finite subset T such that every element of S majorizes some element of T. We define an equivalence relation $\sim$ in T as follows: $I \sim J$ if and only if $(\Omega^k|\alpha|)^I = (\Omega^k|\alpha|)^J$ for any $k \ge 0$, where $|\alpha| = (|\alpha_1|, \dots, |\alpha_n|)$. Let $\{I_1, \dots, I_q\}$ be a complete system of representatives and put

$$f_{\sigma\tau}(k) = \log(\Omega^k|\alpha|)^{I_\sigma} - \log(\Omega^k|\alpha|)^{I_\tau}, \qquad 1 \le \sigma < \tau \le q,$$

$$f_i(k) = -\log|\alpha_i^{(k)}|, \qquad 1 \le i \le n,$$

where $\Omega^k\alpha = (\alpha_1^{(k)}, \dots, \alpha_n^{(k)})$. Then $f_{\sigma\tau}(k)$ and $f_i(k)$ are non-zero linear recurrences of which characteristic roots are all eigenvalues of Ω. Since unity is not an eigenvalue of Ω, none of $f_{\sigma\tau}(k)$, $f_i(k)$ is a polynomial. By Lemma 2 in Kubota [2], $f_{\sigma\tau}(k)$ and $f_i(k)$ respectively have the degree $(\delta_{\sigma\tau}, \theta_{\sigma\tau})$ and (δ_i, θ_i) with $\theta_{\sigma\tau}, \theta_i > 1$. Putting $\theta = \min\{\theta_{\sigma\tau}, \theta_i\}_{1\le\sigma<\tau\le q, 1\le i\le n}$, we have $\theta > 1$. By appealing to Corollary 2 of Lemma 1 in [2], there exists a subset R_0 of $\mathbb{N}$ such that $\delta(R_0) > 0$ and for all $k \in R_0$,

$$|f_{\sigma\tau}(k)|, |f_i(k)| \ge c_2\theta^k, \qquad 1 \le \sigma < \tau \le q,\ 1 \le i \le n.$$

By renumbering the indices of $I_1, \dots, I_q$, there exists a subset R of R_0 such that $\delta(R) > 0$ and for all $k \in R$,

$$\begin{aligned} \log(\Omega^k|\alpha|)^{I_j} - \log(\Omega^k|\alpha|)^{I_{j+1}} &\ge c_2\theta^k, \qquad 1 \le j \le q-1, \\ -\log|\alpha_i^{(k)}| &\ge c_2\theta^k, \qquad 1 \le i \le n. \end{aligned} \tag{1}$$

Let

$$P(z) = \sum_{I\in T, I\sim I_1} c_I z^I.$$

For $I \sim I_1$, we put $I^i = I + (0, \dots, \overset{i}{1}, \dots, 0)$. Then for all large $k \in R$,

$$\begin{aligned} |P(\Omega^k\alpha)| &= |f(\Omega^k\alpha) - P(\Omega^k\alpha)| \\ &\le \sum_{\substack{I\in T\\ I\not\sim I_1}} \sum_{J\ge I} |c_J|\,|(\Omega^k\alpha)^J| + \sum_{\substack{I\in T\\ I\sim I_1}} \sum_{i=1}^{n} \sum_{J\ge I^i} |c_J|\,|(\Omega^k\alpha)^J| \\ &\le \sum_{\substack{I\in T\\ I\not\sim I_1}} c_3(\Omega^k|\alpha|)^I + \sum_{\substack{I\in T\\ I\sim I_1}} \sum_{i=1}^{n} c_3(\Omega^k|\alpha|)^{I^i}. \end{aligned}$$

By the inequality (1), we have

$$\begin{aligned} |P(\Omega^k\alpha)| &\le |T|c_3e^{-c_2\theta^k}(\Omega^k|\alpha|)^{I_1} + n|T|c_3e^{-c_2\theta^k}(\Omega^k|\alpha|)^{I_1} \\ &\le c_4e^{-c_2\theta^k}(\Omega^k|\alpha|)^{I_1}, \end{aligned}$$

for all large $k \in R$. Setting $\beta_i = \alpha_i/|\alpha_i|$ $(1 \le i \le n)$, we have

$$P(\Omega^k\beta) = P(\Omega^k\alpha)/(\Omega^k|\alpha|)^{I_1}$$

and so

$$|P(\Omega^k\beta)| \le c_4e^{-c_2\theta^k}$$

for all large $k \in R$. Applying the lemma to the set R, we obtain a strictly increasing sequence $d_1, d_2, \dots$ as in the lemma. We apply Proposition 2 to the set $S = \{d_1, d_2, \dots\}$ and $P(z)$. Then $r + b_1, \dots, r + b_s \in R$ for infinitely many positive integers r. Therefore

$$\left|\sum_{i=1}^{s} B_i(\Omega^r\beta)P(\Omega^{r+b_i}\beta)\right| \le c_5e^{-c_6\theta^r}$$

for infinitely many r. Then by the equality (ii) in Proposition 2, for some j $(1 \le j \le t)$ and infinitely many r we have

$$|U_j(\Omega^r\beta) - V_j(\Omega^r\beta)| \le c_7e^{-c_6\theta^r/t}. \tag{2}$$

By Baker's theorem [1, Theorem 3.1] for linear forms in logarithms of algebraic numbers,

$$|U_j(\Omega^r\beta)/V_j(\Omega^r\beta) - 1| \geq c_8 e^{-c_9 r} \tag{3}$$

if $U_j(\Omega^r\beta) \neq V_j(\Omega^r\beta)$. By (2) and (3) we conclude that $U_j(\Omega^r\beta) = V_j(\Omega^r\beta)$ for infinitely many r. On the other hand, since $I \sim J \sim I_1$,

$$\begin{aligned} U_j(\Omega^r|\alpha|)/V_j(\Omega^r|\alpha|) &= (\Omega^r|\alpha|)^{(I-J)\Omega^m(\Omega^a-E)} \\ &= (\Omega^{r+m+a}|\alpha|)^{I-J}(\Omega^{r+m}|\alpha|)^{J-I} = 1. \end{aligned}$$

Therefore

$$U_j(\Omega^r\alpha) = V_j(\Omega^r\alpha)$$

for infinitely many r. Applying Proposition 1 to the polynomial $U_j(z) - V_j(z)$, we complete the proof.

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