

INDUCING CHARACTERS AND NILPOTENT SUBGROUPS

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ABSTRACT. If H is a subgroup of a finite group G and $\gamma \in \text{Irr}(H)$ induces irreducibly up to G , we prove that, under certain odd hypothesis, $\mathbf{F}(G)\mathbf{F}(H)$ is a nilpotent subgroup of G .

1. INTRODUCTION

If some character of a subgroup H of a finite group G induces irreducibly up to G , one expects H to be large enough to contain nontrivial information on G . In this note, we relate the Fitting subgroups of H and G .

Theorem A. *Let $H \subseteq G$ and suppose that γ is a character of H with $\gamma^G \in \text{Irr}(G)$. If either $|G : H|$ or $|H : \mathbf{F}(H)|$ is odd, then $\mathbf{F}(G)\mathbf{F}(H)$ is a nilpotent subgroup of G .*

Notice that Theorem A is no longer true without the odd hypothesis, $GL(2, 3)$ being a solvable counterexample (we may take H to be a 3-Sylow normalizer).

Theorem A may be applied to study characters induced from nilpotent subgroups, and this is what we do in Section 3 below.

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2. PROOF OF THEOREM A

The proof of our Theorem A relies on the deep facts on odd fully ramified sections discovered by I. M. Isaacs ([1]).

(2.1) Theorem. *Let $L \subseteq K \triangleleft G$ with $L \triangleleft G$, K/L abelian, and assume that either $|G : K|$ or $|K : L|$ is odd. Let $\phi \in \text{Irr}(L)$ be invariant in G and assume that $\phi^K = e\theta$ for some $\theta \in \text{Irr}(K)$ and integer e . Then there exists $U \subseteq G$ such that*

- (a) $UK = G$ and $U \cap K = L$,
- (b) *If $|K/L| > 1$ and $\xi \in \text{Irr}(U|_\phi)$, then ξ^G is reducible.*

Proof. This is a well known consequence of Theorems (9.1) and (9.2) of [1]. If ξ^G is irreducible, Theorem (9.2) forces the canonical character ψ to be irreducible. However, $\psi\bar{\psi}$ is the permutation character of G on K/L (see the values of the character ψ), and since this action cannot be transitive, $[\psi, \psi] > 1$, a contradiction. \square

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Proof of Theorem A. We argue by double induction, first on $|G|$ and second on $|G : H|$, and we assume that H is proper in G .

First, we show that it is no loss to assume that $G = H\mathbf{F}(G)$. If we write $F = \mathbf{F}(G)$ and $K = HF < G$, since γ^K is irreducible, and either $|K : H|$ or $|H : \mathbf{F}(H)|$ is odd, by the inductive hypothesis, we have that $\mathbf{F}(K)\mathbf{F}(H)$ is nilpotent. But then, since $F \subseteq \mathbf{F}(K)$, the theorem follows in this case.

We claim that we may also assume that H is a maximal subgroup of G . If $H < J < G$, arguing as before, we have that $(F \cap J)\mathbf{F}(H)$ is nilpotent. Since $J = (F \cap J)H$, we have that $(F \cap J)\mathbf{F}(H) \subseteq \mathbf{F}(J)$. Then, we have that either $|G : J|$ or $|J : \mathbf{F}(J)|$ is odd. Now, since $(\gamma^J)^G$ is irreducible, by induction on $|G : H|$, we have that $F\mathbf{F}(H) \subseteq F\mathbf{F}(J)$ is nilpotent, and this proves the claim.

Now, write $L = F \cap H$ and notice that L is normal in G , because $L \triangleleft H$ and $L < \mathbf{N}_F(L)$. Since H is maximal (and F is nilpotent), we deduce that F/L is an abelian p -chief factor of G , for some prime p .

Write $\mathbf{F}(H) = X \times Y$, where X is the Sylow p -subgroup of $\mathbf{F}(H)$ and Y its p -complement and similarly write $F = F_p \times F_{p'}$. Since F/L is a p -group, then $F_p L = F$ and thus $F_p H = G$. Hence, we have that $\mathbf{F}(H)/L$ is a p' -group, because $F_p X$ is a p -group normalized by H and then normal in G .

We now start looking at characters. If $\theta \in \text{Irr}(X)$ is a constituent of γ_X , we claim that θ is G -invariant (recall that $X = F_p \cap L$ is normal in G). Certainly, θ is Y -invariant, since $[X, Y] = 1$ and therefore, the inertia group V of θ in H contains $\mathbf{F}(H)$. Since θ lies under γ , by the Clifford Correspondence there is a character $\delta \in \text{Irr}(V)$ such that $\delta^H = \gamma$. Therefore, δ^G is irreducible, and so it is δ^{FV} . Now, if $V < H$, then FV is proper in G and we apply the inductive hypothesis to conclude that $\mathbf{F}(V)F$ is nilpotent. Since $\mathbf{F}(H) \subseteq \mathbf{F}(V)$, the theorem is proven in this case. So we assume that θ is H -invariant. Now, since H is maximal in G , we have that either θ is G -invariant or its inertia group in G is exactly H . So, if θ is not G -invariant, we have that $I_{F_p}(\theta) = H \cap F_p = X$ and thus, we deduce that θ^{F_p} is irreducible. We now prove that this implies $F\mathbf{F}(H)$ to be nilpotent. First, notice that the p' -group Y acts on the p -group F_p in such a way that $X \subseteq \mathbf{C}_{F_p}(Y)$. Since F_p/X is a chief factor of G , we have that $\mathbf{C}_{F_p}(Y) = X$ or $[F_p, Y] = 1$. Since $F\mathbf{F}(H) = F_p Y$, we may assume that $\mathbf{C}_{F_p}(Y) = X$. Hence, $\mathbf{C}_{F_p/X}(Y) = 1$. But then, the Y -invariant irreducible character θ^{F_p} restricted to X has a unique Y -invariant irreducible constituent, by Problem (13.4) of [2], for instance. Since X is centralized by Y , this implies that θ is stabilized by F_p . Since θ induces irreducibly up to F_p , by Problem (6.1) of [2], we have that $F_p = X$ and thus, that $G = H$, a contradiction. This proves θ to be G -invariant, as claimed.

Now, since $(|FY : F_p|, |F_p : X|) = 1$ and $\mathbf{C}_{F_p/X}(Y) = 1$, we are in the hypotheses of Problem (13.10) of [2]. Hence, we may conclude that there exists a unique Y -invariant $\phi \in \text{Irr}(F_p)$ lying over θ . Since $Y \triangleleft H$, if $h \in H$, notice that ϕ^h is Y -invariant and lies over θ . Therefore, by uniqueness, we have that ϕ is G -invariant. Hence, by Mackey, $(\gamma^G)_{F_p}$ is a multiple of ϕ .

We are now ready to apply the Going Down Theorem (6.18) of [2] and we conclude that $\phi_X = \theta$ or that ϕ is fully ramified over θ . In the first case by Corollary (4.2) of [3], we have that $(\gamma^G)_H$ is irreducible. This is impossible, unless $H = G$.

So, we may assume that ϕ is fully ramified over θ . Now we wish to apply Theorem (2.1) and hence we check that H (up to G -conjugacy) is the unique complement of F_p/X in G . This follows by realizing that $H/X = \mathbf{N}_{G/X}(XY/X)$ together with the

fact that $XY/X = (FY \cap H)/X$ is a Hall p -complement of FY/X . Now, since by our hypotheses we have that either $|F_p : X|$ or $|H : X|$ is odd, we apply Theorem (2.1) to get the final contradiction. \square

(2.3) Corollary. *Suppose that $H \subseteq G$ is nilpotent. If γ is a character of H with γ^G irreducible, then $\mathbf{F}(G)H$ is nilpotent.*

Proof. In this case, $|H : \mathbf{F}(H)|$ is odd and Theorem A applies. \square

3. CHARACTERS INDUCED FROM NILPOTENT SUBGROUPS

In this section, we associate to any $\chi \in \text{Irr}(G)$ a uniquely defined (up to conjugacy in G) pair (S, σ) , where S is a subgroup of G and $\sigma \in \text{Irr}(S)$ induces χ . For solvable groups, we will prove that χ is induced from a nilpotent subgroup if and only if S is nilpotent. In other words, whenever $\chi \in \text{Irr}(G)$ can be obtained via induction from a nilpotent subgroup, this can be done in a standard way.

Before introducing the pair (S, σ) we need to derive a (perhaps) surprising consequence of (2.3).

(3.1) Theorem. *Let G be a solvable group and suppose that $\chi \in \text{Irr}(G)$ is induced from a nilpotent subgroup. If $\chi_{\mathbf{F}(G)}$ is homogeneous, then G is nilpotent.*

Proof. Write $\chi = \gamma^G$, where $\gamma \in \text{Irr}(H)$ and H is maximal with respect to being nilpotent. By Corollary (2.3), we have that $F = \mathbf{F}(G) \subseteq H$. Now, let $M/F = \mathbf{F}(G/F)$ and notice that $M \cap H = F$. This is because $F \subseteq M \cap H \subseteq M$ and hence, $M \cap H$ is both nilpotent and subnormal in G .

Write $\chi_F = e\theta$, where $\theta \in \text{Irr}(F)$ and notice that, in the notation of [2], we have that (G, F, θ) is a character triple. By Theorem (11.28) of [2], we may find (G^*, F^*, θ^*) is an isomorphic character triple with $F^* \subseteq \mathbf{Z}(G^*)$. Therefore, observe that $M^* = \mathbf{F}(G^*)$ (we use the notation $(M/F)^* = M^*/F^*$, where $*$ also denotes the group isomorphism between G/F and G^*/F^*). Since H^* is also nilpotent and $(\gamma^*)^{G^*}$ is irreducible (because γ^G is), we may again apply Corollary (2.3) to conclude that M^*H^* is nilpotent. Hence, the group MH/F is also nilpotent. But in this case, since $F \subseteq H$, we have that H is nilpotent and subnormal in MH . Thus $H \subseteq \mathbf{F}(MH)$ and by the maximality of H , we conclude that $H = \mathbf{F}(MH) \triangleleft MH$. Now, since $M \cap H = F$, we have that $H/F \subseteq \mathbf{C}_{G/F}(M/F)$. Since G is solvable, the group M/F contains its own centralizer and thus, we have that $H/F \subseteq M/F$. This implies $H = F$ and $\theta = \gamma$. Now, since θ is G -invariant and induces irreducibly up to G , we conclude that $F = G$, as required. \square

If G is a finite group and $\chi \in \text{Irr}(G)$, we are going to define a uniquely determined (up to G -conjugacy) pair (S, σ) associated to χ , as follows. Choose $\theta \in \text{Irr}(\mathbf{F}(G))$ to be any irreducible constituent of $\chi_{\mathbf{F}(G)}$, and let $\mu \in \text{Irr}(T|\theta)$ be the Clifford correspondent of χ over θ . If $T = G$, we define $(S, \sigma) = (G, \chi)$. On the other hand, if $T < G$, we inductively define (S, σ) for χ to be the corresponding (S, σ) for μ .

Notice that (S, σ) is determined up to conjugacy in G and that it satisfies: $\mathbf{F}(G) \subseteq S$, $\sigma^G = \chi$ and $\sigma_{\mathbf{F}(S)}$ is homogeneous.

(3.2) Theorem. *Let G be a solvable group and let $\chi \in \text{Irr}(G)$. Then χ is induced from a nilpotent subgroup if and only if S is nilpotent.*

Proof. Certainly, if S is nilpotent, then χ is induced from a nilpotent subgroup so we prove the converse. We argue by induction on $|G|$. Let $F = \mathbf{F}(G)$, let $\theta \in \text{Irr}(F)$

be under χ and let $\mu \in \text{Irr}(T)$ be the Clifford correspondent of χ over θ , so that the pair (S, σ) for μ is also a pair for χ .

Write $\chi = \gamma^G$, where $\gamma \in \text{Irr}(H)$ and H is nilpotent. By Corollary (2.3), we may assume that $F \subseteq H$ and by replacing θ by some G -conjugate, we may assume that γ lies over θ . Now, let $\tau \in \text{Irr}(T \cap H|\theta)$ be the Clifford correspondent of γ over θ and notice that $\tau^G = \chi$. By uniqueness of Clifford correspondents, we have that $\tau^T = \mu$. So we have that μ is also induced from a nilpotent subgroup. Therefore, if T is proper in G , by induction, S is nilpotent and we are done in this case. If $T = G$, by Theorem (3.1), G is nilpotent and the result follows. \square

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