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ASYMPTOTIC BEHAVIOR OF C_0-SEMIGROUPS IN BANACH SPACES

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ABSTRACT. We present optimal estimates for the asymptotic behavior of strongly continuous semigroups $U_A : [0,\infty[\to L(X)$ in terms of growth abscissas of the resolvent function $R(\cdot, A)$ of the generator A. In particular we give Ljapunov's classical stability condition a definite form for (infinite dimensional) abstract Cauchy problems: The abscissa of boundedness of $R(\cdot, A)$ equals the growth bound of the classical solutions of $y' = Ay$.

1. INTRODUCTION

Let $A : X \supseteq D(A) \to X$ denote a closed linear operator on a complex Banach space X. Let $\sigma(A;X)$ and $\rho(A;X) := \mathbb{C}\backslash\sigma(A;X)$ denote the *spectrum* and the *resolvent set* of A, respectively, and $R(\cdot, A) : \rho(A;X) \to L(X), z \mapsto (z\,\mathrm{Id}_x - A)^{-1}$ the *resolvent function*. Recall that A is positive in the sense of Triebel [Tr, p. 91], if

$$\begin{cases} A \text{ is densely defined, }]-\infty, 0] \subseteq \rho(A;X) \text{ and} \\ K := \sup\{(1+|t|)\|R(t,A)\| : t > 0\} < \infty. \end{cases} \tag{1.1}$$

If A is a positive operator and $\alpha \in \mathbb{C}$, then fractional powers A^α exist as densely defined closed linear operators (cf. Triebel [Tr, p. 98] for details).

If $A : X \supseteq D(A) \to X$ generates a C_0-semigroup, then by the Hille-Yosida theorem one can always find $\mu > 0$ such that $\mu - A$ is a positive operator. As the domains $D((\mu - A)^\alpha)$ do not depend on μ as long as $\mu - A$ is positive ([Ko, Theorem 6.4]), the quantities $\omega_\beta(A)$ defined below do not depend on μ.

Let

$$s(A) := \sup\{\mathrm{Re}(z) : z \in \sigma(A;X)\} \tag{1.2}$$

denote the *spectral bound* of A.

We shall subdivide the half-plane $\{z \in \mathbb{C} : \mathrm{Re}\, z \geq s(A)\}$ according to the following abscissas associated with $R(\cdot, A)$ and U_A, respectively: Given $\alpha \in [0,\infty[$, let

$$s_\alpha(A) := \inf\{s > s(A) : \|R(a+ib, A)\| = O(|b|^\alpha) \text{ as } |b| \to \infty \text{ and } \alpha \geq s\} \tag{1.3}$$

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denote the *abscissa of growth order* α for $R(\cdot, A)$ on lines parallel to the imaginary axis. Given $\beta \in \mathbb{C}$, let

$$\begin{cases} \omega_\beta(A) := \sup\{\omega(x) : x \in D((\mu - A)^\beta)\}, \text{ where} \\ \omega(x) := \inf\{\omega \in \mathbb{R} : \lim_{t\to\infty} \|e^{-t\omega} U_A(t)x\| = 0\} \end{cases} \tag{1.4}$$

denote the *growth bound of the semigroup* U_A *on* $D((\mu - A)^\beta)$. The following theorem summarizes the known results on these quantities:

Theorem 1.1. *Suppose* $A : X \supseteq D(A) \to X$ *generates a* C_0*-semigroup* U_A *on a complex Banach space* X*. Then*

(1) (*Slemrod* [Sl])

$$\omega_{m+2}(A) \le s_m(A) \qquad (m \in \mathbb{N} \cup \{0\}). \tag{1.5}$$

(2) (*Weiss* [Wss], *Gearhart* [G]) *If* X *is a Hilbert space, then*

$$\omega_m(A) = s_m(A) \qquad (m \in \mathbb{N} \cup \{0\}). \tag{1.6}$$

(3) (*Wrobel* [Wr]) *If* X *is a* B*-convex Banach space, then*

$$\omega_{m+1}(A) \le s_m(A) \qquad (m \in \mathbb{N} \cup \{0\}), \tag{1.7}$$

and in general (1.7) *cannot be replaced by* (1.6) *outside the class of Hilbert spaces.*

(4) (*van Neerven, Straub and Weis* [vN-S-W])

$$\omega_{1+\varepsilon}(A) \le s_0(A) \qquad (\varepsilon > 0). \tag{1.8}$$

(5) (*van Neerven, Straub and Weis* [vN-S-W]) *If* X *is of Fourier type* p, $1 \le p \le 2$*, then*

$$\omega_\beta(A) \le s_0(A) \qquad \left(\beta > \frac{1}{p} - \frac{1}{q}, \frac{1}{p} + \frac{1}{q} = 1\right). \tag{1.9}$$

(6) (*Weis* [Ws]) *If* X *is Lebesgue-space* $L_p(\Omega, \mu)$ $(1 \le p < \infty)$ *and* U_A *is a semigroup of positive operators, then*

$$\omega_0(A) = s_0(A) = s(A). \tag{1.10}$$

Among other things we shall prove that (1.7) is true for general Banach spaces X. Since the Cauchy problem

$$(AC) \qquad y' = Ay, \quad y(0) = x$$

for $y : [0, \infty[\to X$ has a continuously differentiable solution if and only if $x \in D(A)$, this result shows in particular that—just as in the finite dimensional case—all classical solutions of (AC) have an exponential bound already determined by $s_0(A)$, i.e. by $s(A)$ if U_A is a positive semigroup on a Banach lattice.

For C_0-semigroups U_A we shall prove

$$\omega_{\alpha+1}(A) \le s_{\mathrm{Re}\ \alpha}(A) \qquad (\alpha \in \mathbb{C}, \mathrm{Re}\ \alpha > 0 \text{ or } \alpha = 0). \tag{1.11}$$

If X is a Hilbert space one can prove

$$\omega_\alpha(A) = s_{\mathrm{Re}\ \alpha}(A) \qquad (\alpha \in \mathbb{C}, \mathrm{Re}\ \alpha > 0 \text{ or } \alpha = 0) \tag{1.12}$$

using the Fourier techniques of [Wr], and (1.11) can be improved, if X is B-convex: For a suitable small $\varepsilon > 0$:

$$\omega_{\alpha+1-\varepsilon}(A) \le s_{\mathrm{Re}\ \alpha}(A) \qquad (\alpha \in \mathbb{C}, \mathrm{Re}\ \alpha > 0 \text{ or } \alpha = 0). \tag{1.13}$$

Furthermore we find that (5) holds even for $\beta = \frac{1}{p} - \frac{1}{q}$. Thus (5) fills the gap between our result for general Banach spaces (1.11) and the Hilbert space result (1.12) by showing how these growth estimates depend on the geometry of the underlying Banach space, in particular on its Fourier type. Recall (cf. [P]) that a Banach space has Fourier type p if the Hausdorff-Young inequality for the X-valued Fourier transform F holds, i.e. there exists $C < \infty$ with

$$\|Ff\|_{L_{p'}(X)} \leq C\|f\|_{L_p(X)} \qquad (f \in L_p(X)).$$

In section 4 we show that (1.11) and (1.9) with $\beta = \frac{1}{p} - \frac{1}{q}$ included are best possible in general. More precisely, we show that for $Af(x) := xf'(x)$ in $X := L_p(1,\infty) \cap L_q(1,\infty)$, $1 \leq p \leq 2$, $\frac{1}{p} + \frac{1}{q} = 1$, the function

$$[0,\infty[\, \ni \alpha \mapsto \omega_\alpha(A)$$

satisfies

$$\omega_0(A) = -\frac{1}{q}, \qquad \omega_{\frac{1}{p}-\frac{1}{q}}(A) = \omega_1(A) = -\frac{1}{p} = s(A)$$

and is linear inbetween. Note that p is the Fourier type of X. In particular, for $p = 1$ we have

$$\omega_\alpha(A) \leq s(A) \quad \text{only if } \alpha \geq 1.$$

So (1.11) is best possible, even for positive semigroups.

The paper is organized as follows. Section 2 contains relevant facts on the interpolation of domain spaces of fractional powers. In particular, we show that $\omega_\alpha(A)$ is a convex function of α. Section 3 contains the announced results, whereas section 4 demonstrates that our estimates are best possible.

2. Fractional powers of positive operators and interpolation of domains

We refer the reader to the monograph of Triebel [Tr] and the paper of Komatsu [Ko] for background information on interpolation theory. In this section we collect some relevant facts on domains of fractional powers and prove some basic properties of the growth order function $\alpha \mapsto \omega_\alpha(A)$.

Given an interpolation couple $\{X, Y\}$ of Banach spaces X, Y let

$$(X,Y)_{\Theta,p} \qquad (0 < \Theta < 1,\ 1 \leq p \leq \infty) \tag{2.1}$$

denote the interpolation space corresponding to the real interpolation method ([Tr, p. 24]), and let

$$[X,Y]_\Theta \qquad (0 < \Theta < 1) \tag{2.2}$$

denote the interpolation space corresponding to the complex interpolation method ([Tr, p. 64]).

If $Y \subseteq X$, then

$$(X,Y)_{\widetilde{\Theta},\tilde{q}} \subseteq (X,Y)_{\Theta,q} \tag{2.3}$$

and

$$(X,Y)_{\widetilde{\Theta},1} \subseteq [X,Y]_{\widetilde{\Theta}} \subseteq (X,Y)_{\widetilde{\Theta},\infty} \subseteq [X,Y]_\Theta \tag{2.4}$$

$$(0 < \Theta < \widetilde{\Theta} < 1,\ 1 \leq q,\ \tilde{q} \leq \infty).$$

The following is a consequence of [Tr, 1.15.2]. For the reader's convenience we outline a proof.

Lemma 2.1. *Let $A : X \supseteq D(A) \to X$ denote a positive operator, and let $\alpha, \beta, \gamma \in \mathbb{C}$ such that $0 < \operatorname{Re} \alpha < \operatorname{Re} \gamma < \operatorname{Re} \beta$. Then*

$$(D(A^\alpha), D(A^\beta))_{\frac{\operatorname{Re}(\gamma-\alpha)}{\operatorname{Re}(\beta-\alpha)},1} \subseteq D(A^\gamma) \subseteq (D(A^\alpha), D(A^\beta))_{\frac{\operatorname{Re}(\gamma-\alpha)}{\operatorname{Re}(\beta-\alpha)},\infty} \tag{2.5}$$

and, moreover, for all $\varepsilon > 0$ sufficiently small

$$[D(A^\alpha), D(A^\beta)]_{\frac{\operatorname{Re}(\gamma-\alpha)}{\operatorname{Re}(\beta-\alpha)}+\varepsilon} \subseteq D(A^\gamma) \subseteq [D(A^\alpha), D(A^\beta)]_{\frac{\operatorname{Re}(\gamma-\alpha)}{\operatorname{Re}(\beta-\alpha)}-\varepsilon}. \tag{2.6}$$

Proof. The domain spaces $D(A^\alpha)$ of fractional powers A^α are Banach spaces if equipped with their graph norms ([Tr, p. 99]) and $D(A^\beta) \subseteq D(A^\alpha)$ if $\operatorname{Re} \alpha < \operatorname{Re} \beta$ ([Tr, p. 101]). Thus using (2.3) and (2.4) we see that (2.6) is an immediate consequence of (2.5). In order to prove (2.5) first observe that A^α is a topological isomorphism from $D(A^{\mu+\alpha})$ onto $D(A^\mu)$ if $\operatorname{Re} \mu > 0$ ([Tr, p. 101]). Since the considered interpolation methods are functorial, it suffices to prove

$$(X, D(A^{\beta-\alpha}))_{\frac{\operatorname{Re}(\gamma-\alpha)}{\operatorname{Re}(\beta-\alpha)},1} \subseteq D(A^{\gamma-\alpha}) \subseteq (X, D(A^{\beta-\alpha}))_{\frac{\operatorname{Re}(\gamma-\alpha)}{\operatorname{Re}(\beta-\alpha)},\infty}. \tag{2.7}$$

Consequently we have to consider the following situation: Let

$$0 < \operatorname{Re} \eta < \operatorname{Re} \mu, \quad \varepsilon > 0, \quad 0 < \Theta < 1$$

such that

$$\operatorname{Re} \eta = (\operatorname{Re} \eta + \varepsilon)\Theta.$$

Then by [Tr, Theorem 1.15.2 (f)] we obtain for $m \in \mathbb{N}$, $m > \operatorname{Re} \eta + \varepsilon :$

$$\begin{aligned} (X, D(A^\mu))_{\frac{\operatorname{Re} \eta}{\operatorname{Re} \mu},p} &= (X, D(A^\mu))_{\frac{\operatorname{Re} \eta+\varepsilon}{\operatorname{Re} \mu}\cdot\Theta,p} = (X, D(A^{\eta+\varepsilon}))_{\Theta,p} \\ &= (X, D(A^m))_{\frac{\operatorname{Re} \eta+\varepsilon}{m}\cdot\Theta,p} = (X, D(A^m))_{\frac{\operatorname{Re} \eta}{m},p}. \end{aligned} \tag{2.8}$$

But then [Tr, Theorem 1.15.2 (d)] yields

$$(X, D(A^m))_{\frac{\operatorname{Re} \eta}{m},1} \subseteq D(A^\eta) \subseteq (X, D(A^m))_{\frac{\operatorname{Re} \eta}{m},\infty}. \tag{2.9}$$

Letting $\eta = \gamma - \alpha, \mu = \beta - \alpha$, (2.9) and (2.8) give (2.7).

Corollary 2.2. *Let $A : X \supseteq D(A) \to X$ denote a positive operator, $\alpha \in \mathbb{C}$, $\operatorname{Re} \alpha > 0$, $c, v, w \in \mathbb{R}, v \le w, c > 0$, and let*

$$T : D(A^\alpha) \to L_1([0, \infty[, e^{vt}dt; X)$$

denote a continuous linear operator. If for all $\mu > c > 0$ the operator

$$T : D(A^{\alpha+\mu}) \to L_1([0, \infty[, e^{wt}dt; X)$$

is continuous, then for all β with $\operatorname{Re} \beta = \operatorname{Re} \alpha + c$

$$z_0 := \sup\{r \in \mathbb{R} : T(D(A^\beta)) \subseteq L_1([0, \infty[, e^{rt}dt; X)\} \ge w.$$

Proof. Given $0 < \Theta < 1$, let $z_\Theta := (1 - \Theta)v + \Theta w$. Then by [Tr, Theorem 1.18.5]

$$[L_1([0, \infty[, e^{vt}dt; X), L_1([0, \infty[, e^{wt}dt; X)]_\Theta = L_1([0, \infty[, e^{z_\Theta t}dt; X) \tag{2.10}$$

with equivalent norms.

Let $\varepsilon > 0$ and $0 < \Theta := \frac{c}{(1+\varepsilon)(c+\varepsilon)} < 1$. Then by (2.6) we have

$$D(A^{\alpha+\frac{c}{1+\varepsilon}}) \subseteq [D(A^\alpha), D(A^\beta)]_\Theta \tag{2.11}$$

and (2.10) yields a continuous linear operator

$$T : [D(A^\alpha), D(A^\beta)]_\Theta \to L_1([1,\infty[, e^{z_\Theta t}dt; X). \tag{2.12}$$

Since $D(A^\beta) \subseteq D(A^{\alpha+\frac{c}{1+\varepsilon}})$ for all $\varepsilon > 0$ and (2.10) we get

$$T : D(A^\beta) \to L_1([0,\infty[, e^{z_\Theta t}dt; X)$$

continuous for all $0 < \Theta < 1$.

Consequently $z_0 \geq w$, because $z_0 < w$ implies the existence of $0 < \Theta < 1$ such that $z_0 < z_\Theta < w$, contradicting the definition of z_0.

A similar argument shows that $\alpha \mapsto \omega_\alpha(A)$ is a convex function on $[0,\infty[$.

Corollary 2.3. *Let $A : X \supseteq D(A) \to X$ be the generator of a C_0-semigroup U_A. For $0 \leq \alpha_1 < \alpha_2$, $0 < \Theta < 1$ and $\alpha(\Theta) := (1-\Theta)\alpha_1 + \Theta\alpha_2$ we have*

$$\omega_{\alpha(\Theta)}(A) \leq (1-\Theta)\omega_{\alpha_1}(A) + \Theta\omega_{\alpha_2}(A). \tag{2.13}$$

Proof. We proceed as in the proof of Corollary 2.2. Fix $0 < \Theta < 1$. Then for all $\varepsilon > 0$, we have by (2.6)

$$D((-A)^{\alpha(\Theta)}) \subseteq [D((-A)^{\alpha_1}), D((-A)^{\alpha_2})]_{\Theta-\varepsilon}.$$

Moreover, for all $a_i < -\omega_{\alpha_i}(A)$, we have (cf. (2.10))

$$\begin{aligned}&[L_1([0,\infty[, e^{a_1 t}dt; X), L_1([0,\infty[, e^{a_2 t}dt; X)]_{\Theta-\varepsilon}\\ &\qquad = L_1([0,\infty[, e^{z_{\Theta,\varepsilon}}dt; X), \quad \text{where } z_{\Theta,\varepsilon} = (1-\Theta+\varepsilon)a_1 + (\Theta-\varepsilon)a_2.\end{aligned}$$

Since

$$\begin{aligned}T : D((-A)^{\alpha_i}) &\to L_1([0,\infty[, e^{a_i t}dt; X) \qquad (i = 1,2),\\ x &\mapsto U_A(\cdot)x\end{aligned}$$

are bounded operators, so is

$$T : D((-A)^{\alpha(\Theta)}) \to L_1([0,\infty[, e^{z_{\Theta,\varepsilon} t}dt; X).$$

By the Datko-Pazy Lemma (see Lemma 3.1) we obtain

$$\lim_{t\to\infty} e^{z_{\Theta,\varepsilon} t} U_A(t)x = 0$$

for all $x \in D((-A)^{\alpha(\Theta)})$ and $\varepsilon > 0$.

Consequently, for $x \in D((-A)^{\alpha(\Theta)})$ we have by definition

$$\omega(x) \leq \inf\{-z_{\Theta,\varepsilon} : \varepsilon > 0\} = -a_1(1-\Theta) - \Theta a_2$$

and then

$$\omega_{\alpha(\Theta)}(A) \leq (1-\Theta)(-a_1) + \Theta(-a_2). \tag{2.14}$$

But as (2.14) is true for all a_i such that $a_i < -\omega_{\alpha_i}(A)$ $(i = 1,2)$, we obtain the desired result by taking the infimum over all such a_i.

Remark 2.4. Corollaries 2.2, 2.3 and their proofs as well as (2.5) and (2.6) imply that the growth bounds are the same for all of the usual scales of intermediate spaces. More precisely, if X_α is one of the spaces

$$(X, D(A))_{\alpha,p}, \quad 1 \leq p \leq \infty, \qquad [X, D(A)]_\alpha \qquad \text{or} \qquad D((-A)^\beta)$$

with Re $\beta = \alpha$, then $\omega_\alpha(A)$ is the inf of all w such that for all $x \in X_\alpha$, we have

$$\sup e^{-wt}\|U_A(t)x\| < \infty.$$

3. Asymptotic behavior

We recall the Datko-Pazy Lemma and a consequence of its proof ([Pa, p. 116, Theorem 4.1]).

Lemma 3.1. *Suppose $A : X \supseteq D(A) \to X$ generates a C_0-semigroup U_A and let $1 \le p < \infty$. Then*

(1) *If $x \in X$ is such that $\int_0^\infty \|U_A(t)x\|^p \, dt < \infty$, then $\|U_A(t)x\| \to 0$, i.e. $\omega(x) \le 0$.*
(2) *If for all $x \in X$ one has $\int_0^\infty \|U_A(t)x\|^p \, dt < \infty$, then $\omega_0(A) < 0$.*

We should remark that if we replace the whole space X by $D((-A)^\alpha)$ in (2), we do not get $\omega_\alpha(A) < 0$ but in general only $\omega_\alpha(A) \le 0$ by (1).

The following is our main result.

Theorem 3.2. *Suppose $A : X \supseteq D(A) \to X$ generates a C_0-semigroup U_A. Then for all $\alpha \in \mathbb{C}$ such that* $\operatorname{Re} \alpha > 0$ *we have*

$$\omega_{\alpha+1}(A) \le s_{\operatorname{Re}\,\alpha}(A) \tag{3.1}$$

and

$$\omega_1(A) \le s_0(A). \tag{3.2}$$

Proof. Given $\mu \in \mathbb{R}$, we have $\sigma(A+\mu; X) = \sigma(A; X) + \{\mu\}$ and consequently (1.3) and (1.4) immediately imply that

$$s_{\operatorname{Re}\,\alpha}(A+\mu) = s_{\operatorname{Re}\,\alpha}(A) + \mu$$

and

$$\omega_\alpha(A+\mu) = \omega_\alpha(A) + \mu.$$

So without loss of generality we may assume that $-A$ itself is positive. Furthermore, we can restrict ourselves to a fixed $\alpha \in \mathbb{C}$ with $\operatorname{Re}\,\alpha \ge 0$ and $s_{\operatorname{Re}\,\alpha}(A) < 0$. Observe that for all $\delta > 0$

$$z \mapsto R(z, A) z^{-\alpha}$$

is bounded on $\{z \in \mathbb{C} : \operatorname{Re}\, z \ge s_{\operatorname{Re}\,\alpha}(A) + \delta\}$ by (1.3). Consequently the following is an absolutely convergent integral for $\varepsilon > 0$ and $s_{\operatorname{Re}\,\alpha}(A) + \delta < 0$:

$$I_{\alpha,\varepsilon}(t) := \frac{1}{2\pi i} \int_{\operatorname{Re} z = s_{\operatorname{Re}\,\alpha}(A)+\delta} e^{zt} R(z, A)(-z)^{-\alpha-1-\varepsilon} dz, \tag{3.3}$$

where we consider the principal branch of the fractional power.

Equation (3.3) represents a one-parameter family of bounded operators on X, and a straightforward estimation of (3.3) gives

$$\|I_{\alpha,\varepsilon}(t)\| \le C e^{(s_{\operatorname{Re}\,\alpha}(A)+\delta)t}, \tag{3.4}$$

since $z \mapsto R(z, A)(-z)^{-\operatorname{Re}\,\alpha}$ is bounded along the line $\operatorname{Re}\, z = s_{\operatorname{Re}\,\alpha}(A) + \delta$, and $\int_0^\infty (c+t^2)^{-\frac{1+\varepsilon}{2}} \, dt < \infty$ $(c > 0)$.

Assume for the moment that

$$I_{\alpha,\varepsilon}(t) = U_A(t)(-A)^{-\alpha-1-\varepsilon} \tag{3.5}$$

has been established. Then we apply Corollary 2.2 with $T := U_A$:

$$\begin{aligned}&U_A : X \to L_1([0,\infty[, e^{-(\omega_0(A)+\delta_0)t}dt; X),\\&U_A : D((-A)^{\alpha+1+\varepsilon}) \to L_1([0,\infty[, e^{-(s_{\mathrm{Re}\ \alpha}(A)+\delta_1)t}dt; X)\\&\qquad(\delta_0, \delta_1 > 0 \text{ small}).\end{aligned}$$

Then the conclusion of Corollary 2.2 with $c = 1$ and the Datko-Pazy Lemma 3.1(1) yield

$$-\omega_{\alpha+1}(A) \geq -s_{\mathrm{Re}\ \alpha}(A) - \delta_1$$

and thus

$$\omega_{\alpha+1}(A) \leq s_{\mathrm{Re}\,\alpha}(A)$$

since δ_1 can be chosen arbitrarily small.

So we have to prove (3.5). Since $I_{\alpha,\varepsilon}(t)$ is a bounded operator and $D_\infty(A) := \bigcap_{n\in\mathbb{N}} D(A^n)$ is dense in X, it suffices to prove (3.5) with both sides restricted to $D_\infty(A)$. So let $x \in D_\infty(A)$. By means of the resolvent equation and Cauchy's theorem we obtain with $\Gamma := \{z \in \mathbb{C} : \mathrm{Re}\ z = s_{\mathrm{Re}\ \alpha}(A) + \delta\}$ (just as in the proof of the Dunford calculus)

$$I_{\alpha,\varepsilon}(t)x = \left(\frac{1}{2\pi i}\int_\Gamma (-z)^{-\alpha-1-\varepsilon}R(z,A)\,dz\right)\left(\frac{1}{2\pi i}\int_\Gamma e^{zt}R(z,A)x\,dz\right).$$

Since the second factor on the right-hand side equals $U_A(t)x$, as is well-known (residue theorem), one has to show that for $\mathrm{Re}\ \beta > 0$

$$(-A)^{-\beta} = \frac{1}{2\pi i}\int_\Gamma (-z)^{-\beta}R(z,A)\,dz.$$

But this is routine by deforming the path of integration and taking boundary values on $[0,\infty[$. Indeed, first replace Γ by

$$\begin{aligned}\Gamma_\varepsilon = \,]{+\infty} - i\varepsilon, -i\varepsilon] \cup \left\{z \in \mathbb{C} : |z| = \varepsilon, \frac{\pi}{2} \leq \arg(z) \leq \frac{3}{2}\pi\right\}\\ \cup\, [i\varepsilon, i\varepsilon + \infty[\end{aligned}$$

and observe that by Cauchy's theorem

$$\int_{\Gamma_\varepsilon} (-z)^{-\beta}R(z,A)\,dz = \int_\Gamma (-z)^{-\beta}R(z,A)\,dz;$$

consequently ($0 \in \rho(A;X)$)

$$\begin{aligned}\int_\Gamma (-z)^{-\beta}R(z,A)\,dz &= \lim_{\varepsilon\downarrow 0}\int_0^\infty \{(-t-i\varepsilon)^{-\beta} - (-t+i\varepsilon)^{-\beta}\}R(t,A)\,dt\\&= (e^{i\pi\beta} - e^{-i\pi\beta})\int_0^\infty t^{-\beta}R(t,A)\,dt = 2\pi i(-A)^{-\beta}\end{aligned}$$

(cf. [Tr, p. 98, 1.15.1 (1)] for the last equality).

Remark 3.3. Inequality (3.2) also follows directly from 1.1(4) and (2.3) since the convex function $\alpha \mapsto \omega_\alpha(A)$ is continuous. Furthermore, using the fact that $z \mapsto R(z,A)(-A)^{-\alpha}$ ($0 < \mathrm{Re}\ \alpha$) is bounded on $\{z \in \mathbb{C} : \mathrm{Re}\ z \geq s_{\mathrm{Re}\ \alpha}(A) + \delta\}$ ($\delta > 0$), one can use the techniques of [Wr] and [vN-S-W] to improve Theorem 3.2 as follows:

(1) If X is a Hilbert space, then

$$\omega_\alpha(A) = s_{\mathrm{Re}\ \alpha}(A) \qquad (0 < \mathrm{Re}\ \alpha,\ \text{or}\ \alpha = 0).$$

(2) If X is B-convex, there exists $\varepsilon > 0$ such that

$$\omega_{\alpha+1-\varepsilon}(A) \le s_{\mathrm{Re}\ \alpha}(A) \qquad (0 < \mathrm{Re}\ \alpha,\ \text{or } \alpha = 0).$$

(3) If X has Fourier type p with $1 \le p \le 2$, then with $\frac{1}{p} + \frac{1}{q} = 1$,

$$\omega_{\alpha+\frac{1}{p}-\frac{1}{q}}(A) \le s_{\mathrm{Re}\ \alpha}(A) \qquad (0 < \mathrm{Re}\ \alpha,\ \text{or } \alpha = 0).$$

4. Optimality of the estimates for $\omega(A)$

Consider $X := L_p(1,\infty) \cap L_q(1,\infty)$, $1 \le p \le 2$, $\frac{1}{p} + \frac{1}{q} = 1$, with norm

$$f \mapsto \|f\|_X := \|f\|_p + \|f\|_q$$

and the C_0-semigroup U_A of positive operators (in the lattice sense!) given by

$$(U_A(t)f)(x) = f(e^t x).$$

Then it is well known and easily checked that the generator A is $x\frac{d}{dx}$ on a suitable domain and

$$\omega_0(A) = -\frac{1}{q}, \quad s_0(A) = s(A) = -\frac{1}{p} \quad \text{(see below)}. \tag{4.1}$$

Moreover, since L_p and L_q both have Fourier type p, it is clear that X has Fourier type p.

Fix $f \in C_0^\infty(\mathbb{R})$ with supp $f \subseteq [1,2]$ and for $n \in \mathbb{N}$ let

$$f_n(x) := f(x-n+1).$$

Then $f_n \in C_0^\infty(\mathbb{R}) \subseteq D(A)$ and supp $f_n \subseteq [n, n+1]$. Moreover

$$\|f_n\|_X = \|f_n\|_p + \|f_n\|_q = \|f\|_X \tag{4.2}$$

and

$$\begin{aligned}\|f_n\|_{D(A_p)} &= \|f_n\|_p + \|x \cdot f'(x-n+1)\|_p \\ &\le \|f\|_p + (n+1)\|f'\|_p \le \|f\|_p + (n+1)\|A_p f\|_p \\ &\le (n+1)\|f\|_{D(A_p)}.\end{aligned}$$

Consequently,

$$\|f_n\|_{D(A)} \le (n+1)\|f\|_{D(A)} \tag{4.3}$$

and by [Tr, p. 61, 1.10.1] and [Tr, p. 101]

$$\begin{aligned}\|f_n\|_{D((-A)^\alpha)} &\le c_0 \|f_n\|_X^{1-\alpha} \|f_n\|_{D(A)}^\alpha \\ &\le c(n+1)^\alpha \quad \text{(by (4.2) and (4.3))}\end{aligned} \tag{4.4}$$

with suitable constants c_0, c.

For $t \le \log n$, we have

$$e^{-\frac{t}{q}}\|f_n\|_q + e^{-\frac{t}{p}}\|f_n\|_p \le \|U_A(t) f_n\|_X. \tag{4.5}$$

Suppose we have

$$\|U_A(t) f_n\|_X \le M e^{t\omega} \|f_n\|_{D((-A)^\alpha)}. \tag{4.6}$$

Then taking $t = \log(n)$, (4.4)–(4.6) yield

$$(n)^{-\frac{1}{q}}\|f_n\|_q + n^{-\frac{1}{p}}\|f_n\|_p \le D(n+1)^{\alpha+\omega} \qquad (n \in \mathbb{N})$$

with a suitable constant D, and thus

$$\alpha + \omega \geq -\frac{1}{q}.$$

Since $\omega_\alpha(A) = \inf\{\omega \in \mathbb{R} : \exists M \text{ such that (4.6) holds}\}$, we have

$$-\frac{1}{q} - \alpha \leq \omega_\alpha(A) \tag{4.7}$$

and since X has Fourier type p

$$\omega_\beta(A) \leq s_0(A) \quad \text{for all } \beta > \frac{1}{p} - \frac{1}{q} \text{ (by (1.9))}.$$

Since for all $\beta > \frac{1}{p}$, the functions $f_\beta : [1, \infty[\to \mathbb{R}$, $x \mapsto x^{-\beta}$ are eigenfunctions of A with eigenvalue $-\beta$, $s(A) = s_0(A) = -\frac{1}{p}$ is an accumulation point of eigenvalues, and therefore

$$-\frac{1}{p} \leq \omega_\beta(A) \leq -\frac{1}{p}, \qquad \omega_{\frac{1}{p}-\frac{1}{q}}(A) = -\frac{1}{p}. \tag{4.8}$$

By (2.13) taking $0 < \Theta < 1$ such that $\alpha = \Theta(\frac{1}{p} - \frac{1}{q})$ we obtain

$$\begin{aligned} -\frac{1}{q} - \alpha &\overset{(4.7)}{\leq} \omega_\alpha(A) \leq (1-\Theta)\omega_0(A) + \Theta\omega_{\frac{1}{p}-\frac{1}{q}}(A) \\ &\overset{(4.1),(4.8)}{=} -(1-\Theta)\frac{1}{q} - \Theta\frac{1}{p} = -\frac{1}{q} - \alpha. \end{aligned}$$

Consequently $\alpha \mapsto \omega_\alpha(A)$ fulfills $\omega_0(A) = -\frac{1}{q}$, $\omega_{\frac{1}{p}-\frac{1}{q}} = -\frac{1}{p}$, is linear inbetween and equals $-\frac{1}{p}$ for $\alpha > \frac{1}{p} - \frac{1}{q}$. If $p = 1$, then $\omega_1(A) = s_0(A)$, i.e. (3.1) is best possible in general Banach spaces. If $p = 2$, then $\omega_0(A) = s_0(A)$ as it has to be.

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