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GAPS IN $(\mathcal{P}(\omega), \subset^*)$ AND $(\omega^\omega, \leq^*)$

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Abstract. For a partial order $(P, \leq_P)$, let $\Gamma(P, \leq_P)$ denote the statement that for every $\leq_P$-increasing ω_1-sequence $a \subseteq P$ there is a $\leq_P$-decreasing ω_1-sequence $b \subseteq P$ on top of a such that (a, b) is an (ω_1, ω_1)-gap in P. The main result of this paper is that $\mathfrak{t} > \omega_1 \leftrightarrow \Gamma(\mathcal{P}(\omega), \subset^*) \leftrightarrow \Gamma(\omega^\omega, \leq^*)$. It is also shown, as a corollary, that $\Gamma(\omega^\omega, \leq^*) \rightarrow \mathfrak{b} > \omega_1$ but $\mathfrak{b} > \omega_1 \not\rightarrow \Gamma(\omega^\omega, \leq^*)$.

1. Introduction

If $A, B \in [\omega]^\omega$, then $A \subset^* B$ if and only if $|A \backslash B| < \omega$ and $|B \backslash A| = \omega$. Then the set $\mathcal{P}(\omega)$, the power set of ω, with the relation "$\subset^*$" will be denoted by $(\mathcal{P}(\omega), \subset^*)$. Similarly, there is an ordering relation on ω^ω, the set of all functions from ω to ω, denoted by "$\leq^*$", such that $f \leq^* g$ if and only if $\exists n \forall i (i \geq n \rightarrow f(i) \leq g(i))$ and $f(i) < g(i)$ on an infinite set, where $f, g \in \omega^\omega$. Then the set ω^ω with the relation $\leq^*$ will be denoted by $(\omega^\omega, \leq^*)$. Also, if $f, g \in \omega^\omega$, then $f =^* g$ if and only if $\exists n \forall i (i \geq n \rightarrow f(i) = g(i))$. Let $(P, \leq_P)$ denote either $(\mathcal{P}(\omega), \subset^*)$ or $(\omega^\omega, \leq^*)$. An (ω_1, ω_1)-pregap in P is a pair (a, b) where $a = \langle a_\xi : \xi < \omega_1 \rangle$ and $b = \langle b_\xi : \xi < \omega_1 \rangle$ are subsets of P such that $\forall \xi, \eta < \omega_1 (a_\xi \leq_P b_\eta)$ and $\forall \xi < \eta < \omega_1 (a_\xi \leq_P a_\eta \wedge b_\eta \leq_P b_\xi)$. If there is a $c \in P$ such that $\forall \xi, \eta < \omega_1 (a_\xi \leq_P c \leq_P b_\eta)$, then c splits the pregap (a, b). If no such c exists, then (a, b) is an (ω_1, ω_1)-gap. The relations $\subset^*$ and $\leq^*$ are also used in the definition of cardinals $\mathfrak{b}$ and $\mathfrak{t}$, where

$$\mathfrak{b} = \min\{|\mathcal{B}| : \mathcal{B} \subseteq \omega^\omega, \mathcal{B} \text{ is well ordered by } \leq^* \text{ and } \neg \exists c \in \omega^\omega \forall b \in \mathcal{B} (b \leq^* c)\},$$

$$\begin{aligned} \mathfrak{t} = \min\{|\mathcal{T}| : \mathcal{T} \subseteq \mathcal{P}(\omega), \forall b \in \mathcal{T} (|\omega \backslash b| = \omega),\ &\mathcal{T} \text{ is well ordered by } \subset^* \\ \text{and } \neg \exists c \in \mathcal{P}(\omega)(|\omega \backslash c| = \omega \wedge \forall b \in \mathcal{T} (b \subset^* c))\}&. \end{aligned}$$

These definitions of $\mathfrak{b}$ and $\mathfrak{t}$ are not commonly used in the literature on set theory, but they are the most useful for the analysis here. Other, more familiar and equivalent formulations of $\mathfrak{b}$ and $\mathfrak{t}$ are given in [11].

In [9] it was shown that $MA_{\omega_1} \rightarrow \Gamma(\mathcal{P}(\omega), \subset^*)$. On the other hand, the witnessing sequence for $\mathfrak{t} = \omega_1$ is an example of an $\subset^*$-increasing ω_1-sequence in $\mathcal{P}(\omega)$ which is not a lower half of any (ω_1, ω_1)-gaps in $(\mathcal{P}(\omega), \subset^*)$, so that $\mathfrak{t} = \omega_1 \rightarrow \neg \Gamma(\mathcal{P}(\omega), \subset^*)$. This result motivated Todorčević to conjecture (in a personal conversation) that $\mathfrak{t} > \omega_1$ is equivalent to $\Gamma(\mathcal{P}(\omega), \subset^*)$. Given that a minor modification of the proof of $MA(\omega_1) \rightarrow \Gamma(\mathcal{P}(\omega), \subset^*)$ yields a proof of $MA_{\omega_1} \rightarrow \Gamma(\omega^\omega, \leq^*)$ and given that the witnessing sequence for $\mathfrak{b} = \omega_1$ is an example of an $\leq^*$-increasing

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ω_1-sequence in ω^ω which is not a lower half of any (ω_1, ω_1)-gaps in $(\omega^\omega, \leq^*)$, the conjecture of Todorčević motivates an analogous conjecture about the cardinal $\mathfrak{b}$ and (ω_1, ω_1)-gaps in $(\omega^\omega, \leq^*)$. I denote these two conjectures by

C1: $\mathfrak{t} > \omega_1 \leftrightarrow \Gamma(\mathcal{P}(\omega), \subset^*)$,

C2: $\mathfrak{b} > \omega_1 \leftrightarrow \Gamma(\omega^\omega, \leq^*)$.

There is an interest in resolving these two conjectures. Rothberger [6] proved, for cardinals λ and κ, there is a (λ, κ)-gap in $(\mathcal{P}(\omega), \subset^*)$ if and only if there is a (λ, κ)-gap in $(\omega^\omega, \leq^*)$. This result suggests that $(\mathcal{P}(\omega), \subset^*)$ and $(\omega^\omega, \leq^*)$ have the same properties as far as gaps are concerned. Therefore, if both C1 and C2 were true, then in a model for $\mathfrak{t} = \omega_1 + \mathfrak{b} > \omega_1$, $\gamma(\mathcal{P}(\omega), \subset^*)$ fails and $\Gamma(\omega^\omega, \leq^*)$ holds, thus showing that $(\mathcal{P}(\omega), \subset^*)$ and $(\omega^\omega, \leq^*)$ do not have the same properties with respect to gaps.

The validity of C1 will be established by proving

$$(\diamond) \qquad \mathfrak{t} > \omega_1 \leftrightarrow \Gamma(\mathcal{P}(\omega), \subset^*) \leftrightarrow \Gamma(\omega^\omega, \leq^*).$$

As a corollary of the above equivalence and the existence of a model for $\mathfrak{t} = \omega_1 + \mathfrak{b} > \omega_1$ it follows that $\Gamma(\omega^\omega, \leq^*) \rightarrow \mathfrak{b} > \omega_1$ and $\mathfrak{b} > \omega_1 \not\rightarrow \Gamma(\omega^\omega, \leq^*)$ which settles C2. The implication $\mathfrak{t} > \omega_1 \rightarrow \Gamma(\mathcal{P}(\omega), \subset^*)$ can be restated in the following way: If every $\subset^*$-increasing ω_1-sequence in $\mathcal{P}(\omega)$ has a top (other than ω), then every $\subset^*$-increasing ω_1-sequence in $\mathcal{P}(\omega)$ is a lower half of some (ω_1, ω_1)-gap. However, the assertion that every $\leq^*$-increasing ω_1-sequence in ω^ω has a top is not sufficient to show that every $\leq^*$-increasing ω_1-sequene in ω^ω is a lower half of some (ω_1, ω_1)-gap in ω^ω, which is another way to state $\mathfrak{b} > \omega_1 \not\rightarrow \Gamma(\omega^\omega, \leq^*)$. This further exemplifies differences between cardinals $\mathfrak{b}$ and $\mathfrak{t}$.

The equivalence in $(\diamond)$ can be viewed in another way. Hausdorff [4] showed (in ZFC) that both $(\mathcal{P}(\omega), \subset^*)$ and $(\omega^\omega, \leq^*)$ contain (ω_1, ω_1)-gaps. The result that there is an (ω_1, ω_1)-gap in $(\mathcal{P}(\omega), \subset^*)$ can be stated in the following way: There is an $\subset^*$-increasing ω_1-sequence $a \subseteq \mathcal{P}(\omega)$ and there is a $\subset^*$-decreasing ω_1-sequence $b \subseteq \mathcal{P}(\omega)$ on top of a such that (a, b) forms an (ω_1, ω_1)-gap. The existence of an (ω_1, ω_1)-gap in $(\omega^\omega, \leq^*)$ can be restated in a similar way. Thus, the equivalence in $(\diamond)$ can be seen as a refinement of Hausdorff's results, and this refinement is the best possible.

The equivalence in $(\diamond)$ is the main result of this paper and its proof is given in the next section. For undefined terminology the reader should consult [5].

2.

The equivalence in $(\diamond)$ will be established in the following order: $\mathfrak{t} > \omega_1 \rightarrow \Gamma(\omega^\omega, \leq^*) \rightarrow \Gamma(\mathcal{P}(\omega), \subset^*) \rightarrow \mathfrak{t} > \omega_1$. The hardest implication to establish is the first one and most of this section will be devoted to its proof. Assume $\mathfrak{t} > \omega_1$. Then $\mathfrak{p} > \omega_1$, hence by Bell's Theorem [1] MA_{ω_1} (σ-centered) holds. (Recall that a partial order $(P, \leq_P)$ is σ-centered if $P = \bigcup_{n<\omega} P_n$ and for each n and each $\{p_1, \ldots, p_k\} \subseteq P_n$ there is an $r \in P$ such that $r \leq_P p_1, \ldots, p_k$, i.e. each P_n is centered.) Let $a = \langle a_\xi : \xi < \omega_1 \rangle$ be an $\leq^*$-increasing ω_1-sequence in ω^ω. A $\leq^*$-decreasing ω_1-sequence $b = \langle b_\xi : \xi < \omega_1 \rangle$ on top of a such that (a, b) is an (ω_1, ω_1)-gap in ω^ω will be obtained from an application of MA_{ω_1} (σ-centered) to a suitably defined partial order $\mathbb{P}_a$. In order to guarantee that (a, b) is in fact a gap, the elements of the sequences a and b have to satisfy the following condition:

$$(\star) \quad \forall \xi < \omega_1 \forall i < \omega (a_\xi(i) \leq b_\xi(i)) \wedge \forall \xi, \eta < \omega_1 (\xi < \eta \rightarrow \exists i < \omega (b_\xi(i) < a_\eta(i))).$$

This condition is a variation of the following condition due to Kunen (in unpublished work):

$$\forall \xi < \omega_1 \forall i < \omega (a_\xi(i) \leq b_\xi(i)) \quad \text{and}$$
$$\forall \xi, \eta < \omega_1 (\xi \neq \eta \to \exists i < \omega (a_\xi(i) \not\leq b_\eta(i) \vee a_\eta(i) \not\leq b_\xi(i))).$$

In fact, Hausdorff [3] discovered a condition such that if an (ω_1, ω_1)-pregap in $(\omega^\omega, \leq^*)$ satisfies that condition, then the pregap is in fact an (ω_1, ω_1)-gap. It can be shown (in ZFC) that if an (ω_1, ω_1)-gap in $(\omega^\omega, \leq^*)$ satisfies Hausdorff's condition, then it has an (ω_1, ω_1)-subgap which satisfies $(\star)$ (see [7]). It is easily seen that if (a, b) is an (ω_1, ω_1)-pregap which satisfies $(\star)$, then (a, b) also satisfies Kunen's condition so that (a, b) is in fact a gap. Therefore, the definition of $\mathbb{P}_a$ has to incorporate the requirements in $(\star)$.

Definition 1. Let $a = \langle a_\xi : \xi < \omega_1 \rangle$ be an $\leq^*$-increasing ω_1-sequence in ω^ω.

$$\begin{aligned} \mathbb{P}_a = \{\langle x, y, n, s\rangle : x, y \in [\omega_1]^{<\omega} \wedge n < \omega \wedge s : y \to \omega^n \\ \wedge \forall \xi \in y((\xi \in x \to \forall i < n(a_\xi(i) \leq s(\xi)(i))) \\ \wedge \forall \eta \in x(\eta > \xi \to \exists i < n(s(\xi)(i) < a_\eta(i))))\} \end{aligned}$$

where $\langle x_2, y_2, n_2, s_2\rangle \leq \langle x_1, y_1, n_1, s_1\rangle$ if and only if

(1) $x_1 \subseteq x_2, y_1 \subseteq y_2, n_1 \leq n_2$,
(2) $\forall \xi \in y_1(s_2(\xi) \restriction n_1 = s_1(\xi))$,
(3) $\forall \xi, \eta \in y_1 \forall i < \omega(\xi \leq \eta \wedge n_1 \leq i < n_2 \to s_2(\eta)(i) \leq s_2(\xi)(i))$,
(4) $\forall \xi \in x_1 \forall \eta \in y_1 \forall i < \omega(n_1 \leq i < n_2 \to a_\xi(i) \leq s_2(\eta)(i))$.

Clearly $\mathbb{P}_a$ is a partial order and the next step is to show that $\mathbb{P}_a$ is σ-centered so that MA_{ω_1} (σ-centered) can be applied to it. This will be accomplished by showing that there is a sequence $\langle \mathbb{P}_\alpha : \alpha \leq \omega_1\rangle$ of suborders of $\mathbb{P}_a$ such that $\mathbb{P}_{\omega_1} = \mathbb{P}_a$ and a sequence $\langle i_{\alpha\beta} : \alpha \leq \beta \leq \omega_1\rangle$, with $i_{\alpha\beta} : \mathbb{P}_\alpha \to \mathbb{P}_\beta$, of complete embeddings such that $\forall \alpha, \beta, \gamma(\alpha \leq \beta \leq \gamma \leq \omega_1 \to i_{\alpha\gamma} = i_{\beta\gamma} \circ i_{\alpha\beta})$. Then $\mathbb{P}_\alpha$ can be viewed as a finite support iteration of length ω_1 with σ-centered partial orders, since each $\mathbb{P}_\alpha$, for $\alpha < \omega_1$, will be shown to be σ-centered. This will imply that $\mathbb{P}_a$ is also σ-centered once I show that any finite support iteration of length $\leq \mathfrak{c}$ with σ-centered partial orders is σ-centered.

Definition 2. Let $\mathbb{P}$ and $\mathbb{Q}$ be partial orders. An $i : \mathbb{P} \to \mathbb{Q}$ is a complete embedding if

(a) $\forall p, p' \in \mathbb{P}(p' \leq p \to i(p') \leq i(p))$,
(b) $\forall p, p' \in \mathbb{P}(p' \perp p \leftrightarrow i(p') \perp i(p))$,
(c) $\forall q \in \mathbb{Q} \exists p \in \mathbb{P} \forall p' \in \mathbb{P}(p' \leq p \to i(p') \not\perp q)$.

The symbol "$\perp$" in the above definition denotes the two elements of a partial order are incompatible and compatibility is denoted by $\not\perp$.

For each $\alpha \leq \omega_1$ let $\mathbb{P}_\alpha = \{\langle x, y, n, s\rangle \in \mathbb{P}_a : y \subseteq \alpha\}$ and for each $\alpha \leq \beta \leq \omega_1$ let $i_{\alpha\beta} : \mathbb{P}_\alpha \to \mathbb{P}_\beta$ be the inclusion map $i(p) = p$. Then $\mathbb{P}_a = \mathbb{P}_{\omega_1}$, each $\mathbb{P}_\alpha$ is a suborder of $\mathbb{P}_a$ with the ordering relation inherited from $\mathbb{P}_a$, and $\forall \alpha \leq \beta \leq \gamma \leq \omega_1(i_{\alpha\gamma} = i_{\beta\gamma} \circ i_{\alpha\beta})$.

Lemma 3. *For each $\alpha \leq \beta \leq \omega_1$, $i_{\alpha\beta}$ is a complete embedding.*

Proof. Properties (a) and (b) of Definition 2 are satisfied in a trivial way. For (c), let $q = \langle x_q, y_q, n_q, s_q\rangle \in \mathbb{P}_\beta$. Then $p = \langle x_q, y_q \cap \alpha, n_q, s_q \restriction (y_q \cap \alpha)\rangle$ has the required property. □

Lemma 4. *For each $\alpha < \omega_1$, $\mathbb{P}_\alpha$ is σ-centered.*

Proof. Let $\alpha < \omega_1$ and for each $y \in [\alpha]^{<\omega}$, $n < \omega$, $s \in (\omega^n)^y$ let $\mathbb{P}_{\alpha yns} = \{\langle x, z, m, t\rangle \in \mathbb{P}_\alpha : z = y \wedge m = n \wedge t = s\}$. Then $\mathbb{P}_\alpha = \bigcup\{\mathbb{P}_{\alpha yns} : y \in [\alpha]^{<\omega} \wedge n < \omega \wedge s \in (\omega^n)^y\}$ so that $\mathbb{P}_\alpha$ is a union of countably many suborders. Furthermore, if $\langle x_1, y, n, s\rangle, \ldots, \langle x_k, y, n, s\rangle \in \mathbb{P}_{\alpha yns}$, then $\langle x_1 \cup \cdots \cup x_k, y, n, s\rangle \in \mathbb{P}_{\alpha yns}$ and $\langle x_1 \cup \cdots \cup x_k, y, n, s\rangle \leq \langle x_1, y, n, s\rangle, \ldots, \langle x_k, y, n, s\rangle$. Thus, each $\mathbb{P}_{\alpha yns}$ is centered so that $\mathbb{P}_\alpha$ is σ-centered. □

Therefore, Lemmas 3 and 4 imply that $\mathbb{P}_a$ can be viewed as a finite support iteration of length ω_1 with σ-centered partial orders. Thus, by Proposition 8, $\mathbb{P}_a$ is also σ-centered. The next three lemmas are used in the proof of Proposition 8 below.

Lemma 5. *If $\mathbb{P}$ is a partial order, $\tau_1, \ldots, \tau_n$ are $\mathbb{P}$-names and*

$$p \Vdash \text{“}\exists x \phi(x, \tau_1, \ldots, \tau_n)\text{”},$$

then there is a $\mathbb{P}$-name π such that $p \Vdash$ “$\phi(\pi, \tau_1, \ldots, \tau_n)$”.

Lemma 5 is taken from [5] and will be referred to as the maximal principle. For the next lemma recall that if A is a subset of a partial order $\mathbb{P}$, then A is upward closed if $\forall p \in A \forall q \in \mathbb{P}(p \leq q \rightarrow q \in A)$. It is easily seen that if $A \subseteq \mathbb{P}$ is centered, then there is a $B \subseteq \mathbb{P}$ such that $A \subseteq B$ and B is centered and upward closed.

Lemma 6. *Let $\mathbb{P}$ be a partial order, π a $\mathbb{P}$-name for a partial order and $\mathbf{1} \Vdash$ “π is σ-centered”. Then there are $\mathbb{P}$-names π_n, for $n < \omega$, such that*

$$\mathbf{1} \Vdash \text{“}\pi = \bigcup_{n<\omega} \pi_n \textit{ and each } \pi_n \textit{ is centered and upward closed”}.$$

Proof. Let $\mathbb{P}$ be as above. Since $\mathbf{1} \Vdash$ “π is σ-centered”, then by the maximal principle, choose $\mathbb{P}$-names τ_n, for $n < \omega$, such that

$$(\circ) \qquad \mathbf{1} \Vdash \text{“}\pi = \bigcup_{n<\omega} \tau_n \text{ and each } \tau_n \text{ is centered”}.$$

In addition, since $\mathbf{1} \Vdash$ “τ_n is centered”, then by the remark preceding the lemma, it follows that for each $n < \omega$

$$\mathbf{1} \Vdash \text{“}\exists x(x \subseteq \pi \wedge \tau_n \subseteq x \wedge x \text{ is centered and upward closed)”}.$$

Hence, by the maximal principle again, for each $n < \omega$ there are $\mathbb{P}$-names π_n such that

$$(\bullet) \qquad \mathbf{1} \Vdash \text{“}\tau_n \subseteq \pi_n \subseteq \pi \wedge \pi_n \text{ is centered and upward closed”}.$$

But then $(\circ)$ and $(\bullet)$ imply that

$$\mathbf{1} \Vdash \text{“}\pi = \bigcup_{n<\omega} \pi_n \text{ and each } \pi_n \text{ is centered and upward closed”}$$

which proves the lemma. □

Lemma 7. *Let $p \in \mathbb{P}$, π a $\mathbb{P}$-name for a partial order, $\tau_1, \ldots, \tau_n \in \mathrm{dom}(\pi)$ and*

$$p \Vdash \text{“}\tau_1, \ldots, \tau_n \in \pi \wedge \exists x \in \pi(x \leq_\pi \tau_1, \ldots, \tau_n)\text{”}.$$

Then there is a $q \leq p$ and there is a $\tau \in \mathrm{dom}(\pi)$ such that

$$q \Vdash \text{“}\tau \in \pi \wedge \tau \leq_\pi \tau_1, \ldots, \tau_n\text{”}.$$

Proof. Let $p \in \mathbb{P}$ and $\tau_1, \ldots, \tau_n \in \mathrm{dom}(\pi)$ with

$$p \Vdash \text{“}\tau_1, \ldots, \tau_n \in \pi \wedge \exists x \in \pi (x \leq_\pi \tau_1, \ldots, \tau_n)\text{”}.$$

Then there is a $\mathbb{P}$-name σ and an $r \leq p$ such that $r \Vdash$ “$\sigma \in \pi \wedge \sigma \leq_\pi \tau_1, \ldots, \tau_n$”. Let G be a $\mathbb{P}$-generic filter with $r \in G$. Then since $\sigma_G \in \pi_G$ there is a $\tau \in \mathrm{dom}(\pi)$ such that $\sigma_G = \tau_G$. But now, any statement which is true in the generic extension by G is forced to be true by some $r' \in G$, i.e. $r' \Vdash$ “$\sigma = \tau$”. Finally, since $r, r' \in G$ there is a $q \in G$, with $q \leq r, r'$, such that $q \Vdash$ “$\tau \in \pi \wedge \tau \leq_\pi \tau_1, \ldots, \tau_n$”. But this proves the lemma since also $q \leq p$. □

Proposition 8. *For $\lambda \leq \mathfrak{c}$ let $\langle\langle \mathbb{P}_\alpha : \alpha \leq \lambda\rangle, \langle \pi_\alpha : \alpha < \lambda\rangle\rangle$ be a finite support iteration such that $\mathbf{1}_{\mathbb{P}_\alpha} \Vdash$ “π_α is σ-centered” for each $\alpha < \lambda$. Then $\mathbb{P}_\lambda$ is σ-centered.*

Proof. I may assume that $\lambda = \mathfrak{c}$ since otherwise I can extend the iteration above with the trivial partial order to obtain a finite support iteration of length $\mathfrak{c}$. By Lemma 6, choose $\mathbb{P}_\alpha$-names π_α^n such that

$$\mathbf{1}_{\mathbb{P}_\alpha} \Vdash \text{“}\pi_\alpha = \bigcup_{n<\omega} \pi_\alpha^n \text{ and each } \pi_\alpha^n \text{ is centered and upward closed”}.$$

Let $\mathcal{B}$ be a countable base for 2^ω (the set of all functions from ω to 2) and let $\langle f_\xi : \xi < \mathfrak{c}\rangle$ be a 1-1 enumeration of 2^ω. For each $B \in \mathcal{B}$ let $B' = \{\alpha : f_\alpha \in B\}$. Let $\mathcal{T} = \{T \in [\mathcal{B}]^{<\omega} : \forall A, B \in T (A \cap B = \varnothing)\}$. Let $\mathcal{A}$ be the set of all functions $F : \mathfrak{c} \to \omega$ such that there is a $\{B_0, \ldots, B_n\} \in \mathcal{T}$ and F is constant on each B_i' and on $\mathfrak{c} \setminus \bigcup_{i \leq n} B_i'$. Then $\mathcal{A}$ is countable and for each $F \in \mathcal{A}$ let

$$\mathbb{Q}_F = \{p \in \mathbb{P}_\mathfrak{c} : \forall \alpha < \mathfrak{c} (p \restriction \alpha \Vdash_\alpha \text{“}p(\alpha) \in \pi_\alpha^{F(\alpha)}\text{”})\}.$$

Claim 1. $\bigcup_{F \in \mathcal{A}} \mathbb{Q}_F$ is dense in $\mathbb{P}_\mathfrak{c}$.

Proof. For $\alpha \leq \mathfrak{c}$ let $\mathbb{Q}_\alpha = \{p \in \bigcup_{F \in \mathcal{A}} \mathbb{Q}_F : \mathrm{supp}(p) \subseteq \alpha\}$ and $\mathbb{P}_\mathfrak{c} \restriction \alpha = \{p \in \mathbb{P}_\mathfrak{c} : \mathrm{supp}(p) \subseteq \alpha\}$. It suffices to prove, by induction, that each $\mathbb{Q}_\alpha$ is dense in $\mathbb{P}_\mathfrak{c} \restriction \alpha$. The initial step of the induction is trivial since $\mathbb{P}_0$ is σ-centered and no forcing is involved at this stage. Fix α and assume that $\mathbb{Q}_\beta$ is dense in $\mathbb{P}_\mathfrak{c} \restriction \beta$ for each $\beta < \alpha$. If α is a limit ordinal, since the iteration is with finite supports, the result follows by the induction hypothesis. Now assume $\alpha = \beta + 1$ and let $p \in \mathbb{P}_\mathfrak{c} \restriction \alpha$. I may assume that $\beta \in \mathrm{supp}(p)$ since otherwise the conclusion follows by the induction hypothesis. Then $p \restriction \beta \in \mathbb{P}_\mathfrak{c} \restriction \beta$ and $p \restriction \beta \Vdash_\beta$ “$p(\beta) \in \bigcup_{n<\omega} \pi_\beta^n$”. Let $p' \in \mathbb{P}_\mathfrak{c} \restriction \beta$ and $n < \omega$ be such that $p' \leq p \restriction \beta$ and $p' \Vdash_\beta$ “$p(\beta) \in \pi_\beta^n$”. Since $\mathbb{Q}_\beta$ is dense in $\mathbb{P}_\mathfrak{c} \restriction \beta$ let $q \in \mathbb{Q}_\beta$ such that $q \leq p'$ and define r as follows:

$$r(\xi) = \begin{cases} q(\xi), & \text{if } \xi \neq \beta, \\ p(\xi), & \text{if } \xi = \beta. \end{cases}$$

Then $r \in \mathbb{P}_\mathfrak{c}$ and $r \leq p$. In fact $r \in \mathbb{Q}_\alpha$, as I will show next. Let $\{\alpha_0, \ldots, \alpha_k\}$ enumerate $\mathrm{supp}(r)$ in increasing order. Note that $\alpha_k = \beta$. Then the functions $f_{\alpha_0}, \ldots, f_{\alpha_k}$ are all distinct so that there are pairwise disjoint $B_{\alpha_0}, \ldots, B_{\alpha_k} \in \mathcal{B}$ such that $f_{\alpha_i} \in B_{\alpha_i}$, for $i \leq k$. By the definition of r, for each $i \leq k$, choose n_i such that $r \restriction \alpha_i \Vdash_{\alpha_i}$ “$r(\alpha_i) \in \pi_{\alpha_i}^{n_i}$” and define $F : \mathfrak{c} \to \omega$ by

$$F(\alpha) = \begin{cases} n_i, & \text{if } \exists i \leq k (f_\alpha \in B_{\alpha_i}), \\ 0, & \text{if } f_\alpha \in 2^\omega \setminus \bigcup_{i \leq k} B_{\alpha_i}. \end{cases}$$

Then $F \in \mathcal{A}$ and since

$$\forall \alpha < \mathfrak{c}(\mathbf{1}_{\mathbb{P}_\alpha} \Vdash \text{“}\forall \check{n} < \check{\omega}(\pi_\alpha^n \text{ is upward closed)”})$$

it follows that $r \in \mathbb{Q}_F$. In fact, $r \in \mathbb{Q}_\alpha$ since $\forall \alpha > \beta(r(\alpha) = \mathbf{1}_\alpha)$ and the Claim is proved. □

Claim 2. $\forall F \in \mathcal{A}(\mathbb{Q}_F$ is centered).

Proof. Let $F \in \mathcal{A}$, $p_0, \ldots, p_k \in \mathbb{Q}_F$ and let $\{\alpha_0, \ldots, \alpha_l\}$ be an enumeration of $\bigcup_{i \leq k} \operatorname{supp}(p_i)$ in increasing order. Then $\forall i, j \leq k(p_i \restriction \alpha_0 = p_j \restriction \alpha_0)$ so let $q_0 = p_0 \restriction \alpha_0$ and note that $\forall i \leq k(q_0 \Vdash_{\alpha_0} \text{“}p_i(\alpha_0) \in \pi_{\alpha_0}^{F(\alpha_0)}\text{”})$. Since also $q_0 \Vdash_\alpha \text{“}\pi_{\alpha_0}^{F(\alpha_0)}$ is centered”, it follows by Lemma 7 that there is a $q_0' \leq q_0$, with $q_0' \in \mathbb{P}_{\alpha_0}$, and a $\tau_0 \in \operatorname{dom}(\pi_{\alpha_0})$ such that

$$q_0' \Vdash_{\alpha_0} \text{“}\tau_0 \pi_{\alpha_0} \wedge \forall \check{i} \leq \check{k}(\tau_0 \leq p_i(\alpha_0))\text{”}.$$

Now define q_1 by

$$q_1(\xi) = \begin{cases} q_0'(\xi), & \text{if } \xi < \alpha_0, \\ \tau_0, & \text{if } \xi = \alpha_0, \\ \mathbf{1}(\xi), & \text{if } \alpha_0 < \xi < \alpha_1. \end{cases}$$

Then $q_1 \in \mathbb{P}_{\alpha_1}$ and

$$\forall i \leq k(q_1 \leq p_i \restriction \alpha_1 \wedge q_1 \Vdash_{\alpha_1} \text{“}p_i(\alpha_1) \in \pi_{\alpha_1}^{F(\alpha_1)}\text{”}).$$

Since also $q_1 \Vdash_{\alpha_1} \text{“}\pi_{\alpha_1}^{F(\alpha_1)}$ is centered”, by Lemma 7 again, there is a $q_1' \leq q_1$, with $q_1' \in \mathbb{P}_{\alpha_1}$, and a $\tau_1 \in \operatorname{dom}(\pi_{\alpha_1})$ such that

$$q_1' \Vdash_{\alpha_1} \text{“}\tau_1 \in \pi_{\alpha_1} \wedge \forall \check{i} \leq \check{k}(\tau_1 \leq p_i(\alpha_1))\text{”}.$$

Now, repeat this process l times. It is clear that after l steps there is a q_l and a $\tau_l \in \operatorname{dom}(\pi_{\alpha_l})$ such that $q_l \in \mathbb{P}_{\alpha_l}, \forall i \leq k(q_l \leq p \restriction \alpha_l)$ and

$$q_l \Vdash_{\alpha_l} \text{“}\tau_l \in \pi_{\alpha_l} \wedge \forall \check{i} \leq \check{k}(\tau_l \leq p_i(\alpha_l))\text{”}.$$

Finally define q by

$$q(\xi) = \begin{cases} q_l(\xi), & \text{if } \xi < \alpha_l, \\ \tau_l, & \text{if } \xi = \alpha_l, \\ \mathbf{1}(\xi), & \text{if } \alpha_l < \xi < \mathfrak{c}. \end{cases}$$

Then $q \in \mathbb{P}_\mathfrak{c}$ and $\forall i \leq k(q \leq p_i)$ so that $\mathbb{Q}_F$ is centered. This proves the second Claim.

Now let $\mathbb{P}_F = \{p \in \mathbb{P}_\mathfrak{c} : \exists q \in \mathbb{Q}_F(q \leq p)\}$ for each $F \in \mathcal{A}$. Then Claim 1 implies that $\mathbb{P}_\mathfrak{c} = \bigcup_{F \in \mathcal{A}} \mathbb{P}_F$. In addition, by Claim 2, each $\mathbb{P}_F$ is centered. Hence $\mathbb{P}_\mathfrak{c}$ is σ-centered and the proof of the Proposition is complete. □

It is important to note that a finite support iteration of length $\mathfrak{c}^+$ with any partial order with at least two incompatible elements leads to the partial order which is not σ-centered. The proof of this is analogous to the proof that a Tychonoff product of $\mathfrak{c}^+$ copies of the two point discrete space is not separable.

At this point I can conclude that $\mathbb{P}_a$ is σ-centered, and this completes all of the analysis necessary to prove the main result.

Theorem 9. *The following are equivalent.*

(i) $\mathfrak{t} > \omega_1$.

(ii) $\Gamma(\mathcal{P}(\omega), \subset^*)$.

(iii) $\Gamma(\omega^\omega, \leq^*)$.

Proof. [(i) → (iii)] Let $\mathfrak{t} > \omega_1$. Then MA_{ω_1} (σ-centered) holds. Let $a = \langle a_\xi : \xi < \omega_1 \rangle$ be an $\leq^*$-increasing ω_1-sequence in ω^ω. Then by Proposition 8, $\mathbb{P}_a$ is σ-centered. Let G be a filter in $\mathbb{P}_a$ and for each $\eta < \omega_1$ let

$$b_\eta = \bigcup \{ s(\eta) : \exists p \in G (p = \langle x_p, y_p, n_p, s_p \rangle \wedge s = s_p) \}.$$

Condition (4) of Definition 1 together with the requirement that for each $\xi, \eta < \omega_1$ and each $m < \omega$ the filter G has a nonempty intersection with the dense sets

$$D_{\xi\eta m} = \{ \langle x, y, n, s \rangle \in \mathbb{P}_a : \xi \in x \wedge \eta \in y \wedge n \geq m \}$$

will guarantee that $\forall \xi, \eta < \omega_1 (a_\xi \leq^* b_\eta)$. In addition, condition (3) of Definition 1 together with the requirement that for each $\xi < \eta < \omega_1$ and each $m < \omega$ the filter G has a nonempty intersection with the following dense sets

$$E_{\xi\eta m} = \{ \langle x, y, n, s \rangle \in \mathbb{P}_a : \xi, \eta \in y \wedge |\{ i : s(\eta)(i) < s(\xi)(i) \}| \geq m \}$$

will guarantee that $\forall \xi < \eta < \omega_1 (b_\eta \leq^* b_\xi)$. Therefore, to satisfy the requirements that $\forall \xi, \eta < \omega_1 (a_\xi \leq^* b_\eta)$ and $\forall \xi < \eta < \omega_1 (b_\eta \leq^* b_\xi)$ the filter G needs to intersect ω_1 dense subsets of $\mathbb{P}_a$ and MA_{ω_1} (σ-centered) guarantees that there is one such filter. In addition, the definition of $\mathbb{P}_a$ implies that $\forall \xi < \omega_1 \forall i < \omega (a_\xi(i) \leq b_\xi(i))$ and $\forall \xi, \eta < \omega_1 (\xi < \eta \to \exists i < \omega (b_\xi(i) < a_\eta(i)))$ so that (a, b) is in fact an (ω_1, ω_1)-gap in ω^ω.

[(iii)→(ii)] Let $a = \langle a_\xi : \xi < \omega_1 \rangle$ be an $\subset^*$-increasing ω_1-sequence in $\mathcal{P}(\omega)$ and let $\chi = \langle \chi_\xi : \xi < \omega_1 \rangle$ be the sequence of corresponding characteristic functions. Note that χ is an $\leq^*$-increasing ω_1-sequence in ω^ω. Then $\Gamma(\omega^\omega, \leq^*)$ implies that there is a $\leq^*$-decreasing ω_1-sequence $\psi = \langle \psi_\xi : \xi < \omega_1 \rangle$ on top of χ such that (χ, ψ) is an (ω_1, ω_1)-gap. For each $\xi < \omega_1$ define functions $\phi_\xi : \omega \to 2$ as follows:

$$\phi_\xi(n) = \begin{cases} 1, & \text{if } \psi_\xi(n) \geq 1, \\ 0, & \text{if } \psi_\xi(n) = 0. \end{cases}$$

Since $\forall \xi < \omega_1 \forall n < \omega (\chi_\xi(n) \leq 1)$ and since $\forall \xi, \eta < \omega_1 (\chi_\xi \leq^* \psi_\eta)$ it follows that $\forall \xi, \eta < \omega_1 (\chi_\xi \leq^* \phi_\eta)$. In addition, if $\exists \xi < \omega_1 \forall \zeta < \omega_1 (\xi < \zeta \to \phi_\xi =^* \phi_\zeta)$, then for some large enough $\xi < \omega_1$, ϕ_ξ splits (χ, ψ), which contradicts the fact that (χ, ψ) is a gap. Therefore, one can choose inductively an $A \in [\omega_1]^{\omega_1}$ such that $\langle \phi_\xi : \xi \in A \rangle$ is a $\leq^*$-decreasing ω_1-sequence and $\langle \chi_\xi, \phi_\eta : \xi < \omega_1, \eta \in A \rangle$ is an (ω_1, ω_1)-gap in $(\omega^\omega, \leq^*)$. Let $\{ \alpha_\xi : \xi < \omega_1 \}$ be an increasing enumeration of A and let b_ξ be the subset of ω whose characteristic function is ϕ_{α_ξ}. Then $\langle a_\xi, b_\xi : \xi < \omega_1 \rangle$ is an $(\omega_1, \omega - 1)$-gap in $(\mathcal{P}(\omega), \subset^*)$ so that $\Gamma(\mathcal{P}(\omega), \subset^*)$ holds.

[(ii)→(i)] The definition of $\mathfrak{t}$ makes the proof of this part trivial. □

It follows trivially from the definitions that $\Gamma(\omega^\omega, \leq^*) \to \mathfrak{b} > \omega_1$.

Finally, let M be a model for ZFC in which $\mathfrak{t} = \omega_1$ and $\mathfrak{b} > \omega_1$. Such a model exists by the result of Solomon [8]. Then by Theorem 9, $\Gamma(\omega^\omega, \leq^*)$ fails in M so that $\mathfrak{b} > \omega_1 \not\to \Gamma(\omega^\omega, \leq^*)$, which settles C2. To be more explicit, in M, let $a = \langle a_\xi : \xi < \omega_1 \rangle$ be an $\subset^*$-increasing ω_1-sequence in $\mathcal{P}(\omega)$ which is a witness for $\mathfrak{t} = \omega_1$. Let $\chi = \langle \chi_\xi : \xi < \omega_1 \rangle$ be the sequence of corresponding characteristic functions in ω^ω. Then χ is not a lower half of any (ω_1, ω_1)-gaps in $(\omega^\omega, \leq^*)$. To see this, by way of contradiction, let $\psi = \langle \psi_\xi : \xi < \omega_1 \rangle$ be a $\leq^*$-decreasing ω_1-sequence on top of χ such that (χ, ψ) is an (ω_1, ω_1)-gap. Since the unit function, $e(n) = 1$ for each n, does not split (χ, ψ) there is a $\xi_0 < \omega_1$ such that $\{i : \psi_{\xi_0}(i) < e(i)\}$ is infinite. Let $c = \{i : \psi_{\xi_0}(i) \geq 1\}$. Then $|\omega \backslash c| = \omega$ and $\forall \xi < \omega_1 (a_\xi \subset^* c)$, which is a contradiction since a is a witness for $\mathfrak{t} = \omega_1$.

3.

Let $(a, b) = \langle a_\xi, b_\xi : \xi < \omega_1 \rangle$ and $(c, d) = \langle c_\xi, d_\xi : \xi < \omega_1 \rangle$ be two (ω_1, ω_1)-gaps in ω^ω. Then (a, b) and (c, d) are equivalent if $\forall \xi < \omega_1 \exists \eta < \omega_1 (a_\xi \leq^* c_\eta \wedge d_\eta \leq^* b_\xi)$ and $\forall \xi < \omega_1 \exists \eta < \omega_1 (c_\xi \leq^* a_\eta \wedge b_\eta \leq^* d_\xi)$. The gap (a, b) is disjoint if $\forall \xi < \omega_1 \forall i < \omega (a_\xi(i) \leq b_\xi(i))$. Elements of each (ω_1, ω_1)-gap can be changed a finite amount to obtain an equivalent and a disjoint gap. The gap constructed in the proof of [(i)→(iii)] of Theorem 9 is a disjoint gap. The fact that the gap in that proof was made to satisfy $(\star)$ is not coincidental. In the presence of MA, all (ω_1, ω_1)-gaps are equivalent to gaps which satisfy condition $(\star)$, as it is implicitly stated in the next proposition.

Proposition 10. *Assume MA and let $(a, b) = \langle a_\xi, b_\xi : \xi < \omega_1 \rangle$ be an (ω_1, ω_1)-gap in ω^ω. Then there is a $B \in [\omega_1]^{\omega_1}$ such that $\forall \xi, \eta \in B(\xi < \eta \rightarrow \exists i < \omega(b_\xi(i) < a_\eta(i)))$.*

Proof. Let $(a, b) = \langle a_\xi, b_\xi : \xi < \omega_1 \rangle$ be an (ω_1, ω_1)-gap in ω^ω and define a partial order $\mathbb{Q}$ as follows:

$$\mathbb{Q} = \{x \in [\omega_1]^{<\omega} : \forall \xi, \eta \in x(\xi < \eta \rightarrow \exists i < \omega(b_\xi(i) < a_\eta(i)))\}$$

where $x_2 \leq x$, if and only if $x_2 \supseteq x_1$. Then $\mathbb{Q}$ is indeed a partial order and to show that $\mathbb{Q}$ has the countable chain condition, by way of contradiction, assume that $A = \{x_\xi : \xi < \omega_1\}$ is an uncountable antichain in $\mathbb{Q}$. By the Δ-system lemma I may assume that A forms a Δ-system with root r. In fact, since $\{x_\xi \backslash r : \xi < \omega_1\}$ is also an uncountable antichain in $\mathbb{Q}$ I may assume that $r = \varnothing$. By another thinning process I may also assume that $\forall \xi, \eta < \omega_1 (\xi < \eta \rightarrow \max(x_\xi) < \min(x_\eta))$. Now, for each $\xi < \omega_1$ let m_ξ be such that

$$\forall \eta, \theta \in x_\xi \forall i \geq m_\xi (a_\eta(i) \leq b_\theta(i)) \quad \text{and}$$
$$\forall \eta, \theta \in x_\xi \forall i \geq m_\xi (\eta \leq \theta \rightarrow a_\eta(i) \leq a_\theta(i) \wedge b_\theta(i) \leq b_\eta(i)).$$

Then by a final thinning process I may assume that $m_\xi = m$ for some $m < \omega$ and all $\xi < \omega_1$. But now, since A is an uncountable antichain it follows that

$$\forall \alpha, \beta < \omega_1 (\alpha < \beta \rightarrow \exists \xi \in x_\alpha \exists \eta \in x_\beta \forall i < \omega (a_\eta(i) \leq b_\xi(i))).$$

But his in turn implies that if $\alpha_\xi = \min(x_\xi)$, then

$$(\triangleright) \qquad \forall \xi, \eta < \omega_1 (\xi < \eta \rightarrow \forall i \geq m(a_{\alpha_\eta}(i) \leq b_{\alpha_\xi}(i))).$$

Now I use $(\triangleright)$ to obtain a contradiction. Define $c : \omega \rightarrow \omega$ as follows: If $i < m$, then $c(i) = 0$, and if $i \geq m$, then let n be the maximal integer such that there is an uncountable set $C \subseteq \omega_1$ such that $\forall \xi \in C(a_{\alpha_\xi}(i) = n)$ and let $c(i) = n$. Condition $(\triangleright)$ implies that c is well defined. I claim that $\forall \xi < \omega_1 (a_{\alpha_\xi} \leq^* c \leq^* b_{\alpha_\xi})$. To show that $\forall \xi < \omega_1 (a_{\alpha_\xi} \leq^* c)$, by way of contradiction, assume that $D = \{i : c(i) < a_{\alpha_\theta}\}$ is infinite for some $\theta < \omega_1$. For each $\eta > \theta$ let k_η be such that $\forall i \geq k_\eta (a_{\alpha_\theta}(i) \leq a_{\alpha_\eta}(i))$. Choose $E \in [\omega_1 \backslash \theta]^{\omega_1}$ and a $k < \omega$ such that $\forall \eta \in E(k_\eta = k)$. Since D is infinite choose $i \in D$ with $i > \max(m, k)$ and an uncountable $F \subseteq E$ such that for some $l < \omega$ it follows that $\forall \eta \in F(a_{\alpha_\eta}(i) = l)$. Now by the definition of c it follows that $c(i) < a_{\alpha_\theta}(i) \leq l \leq c(i)$, which is a contradiction. Hence, $\forall \xi < \omega_1(a_{\alpha_\xi} \leq^* c)$. The proof of $\forall \xi < \omega_1(c \leq^* b_{\alpha_\xi})$ is analogous and I omit it. Therefore c splits $\langle a_{\alpha_\xi}, b_{\alpha_\xi} : \xi < \omega_1 \rangle$. But this implies that c also splits (a, b) since $\langle a_{\alpha_\xi}, b_{\alpha_\xi} : \xi < \omega_1 \rangle$ and (a, b) are equivalent. This is a contradiction by the assumption that (a, b) is a gap. Therefore $\mathbb{Q}$ has the countable chain condition.

Now, MA implies that there is an uncountable $K \subseteq \mathbb{Q}$ of pairwise compatible elements. At this point it is easy to see that if $B = \bigcup K$, then

$$\forall \xi, \eta \in B(\xi < \eta \rightarrow \exists i < \omega(b_\xi(i) < a_\eta(i)))$$

and the proof of the proposition is complete. □

In [2] and again in [10] it was shown that, in the presence of MA, every disjoint (ω_1, ω_1)-gap in $(\omega^\omega, \leq^*)$ has an (ω_1, ω_1)-subgap which satisfies Kunen's condition. Thus, Proposition 10 is a further refinement of this result and, together with the preceding remarks, it states that, in the presence of MA, all (ω_1, ω_1)-gaps are gaps for essentially the same reason, namely, that each gap is equivalent to a gap which satisfies $(\star)$.

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