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THEOREM OF KURATOWSKI-SUSLIN FOR MEASURABLE MAPPINGS. II

ANDRZEJ WIŚNIEWSKI

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Abstract. The purpose of this paper is to describe these μ-measurable mappings on a separable complete metric space with the Borel measure μ, which transform every μ-measurable set onto a μ-measurable one. The obtained results are a generalization of the classical outcomes of Suslin and Kuratowski and the results from our previous paper.

1. Introduction

Let X be a separable complete metric space and let f be a one-to-one mapping on X. In his paper [4] Suslin proved that if f is a continuous function, then for every Borel subset B of X the image $f(B)$ is also a Borel set. Kuratowski in [1] extended this theorem to the case of Borel mappings. Namely, he proved the following theorem.

Theorem of Kuratowski. *Let X be a separable complete metric space and X_1 a Borel subset of X. If f is a one-to-one Borel measurable mapping from X_1 into X, then $f(B)$ is a Borel set for every Borel subset B of X_1.*

For details concerning the above theorem see also [2] (Chapter I.4).

In our previous paper [5] we investigated the possibility of the generalization of the above theorem of Suslin-Kuratowski to the case of measurability (of sets B and $f(B)$) with respect to some measure on X instead of their Borel measurability. It appears that such a generalization need not always be true, even in the case of the measurability with respect to the Lebesgue measure on a real line (see [5], Example 1) or in the case of translations in a linear space (see [5], Example 2). In [5] we have given the conditions under which the above mentioned generalization of the theorem of Suslin-Kuratowski is possible. Namely, it was shown that a one-to-one Borel mapping f on X transforms every measurable set (with respect to some measure μ on X) onto a measurable one if and only if the measure μ is absolutely continuous with respect to the measure μ_f (an image of μ under the mapping f) ([5], Theorem 1).

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In the present paper we shall continue this investigation. Our purpose will be the further extension of the theorem of Suslin-Kuratowski. We generalize this theorem to the case of μ-measurable mappings. Similarly as in the case of Borel mappings we give also the description of these μ-measurable mappings on X for which such a generalization is possible. Moreover, we prove some similar properties of μ-measurable mappings, which may be treated as a specific form of the theorem of Suslin-Kuratowski, however concerning not images but inverse images of measurable sets.

Let $(X, \mathcal{B}, \mu)$ be a measure space and $(Y, \mathcal{F})$ a measurable space. Let f be a measurable mapping from X into Y. By μ_f we shall denote the image of μ under the mapping f, i.e. a measure on $\mathcal{F}$ defined by the formula $\mu_f(A) = \mu(f^{-1}(A))$ for $A \in \mathcal{F}$.

A measure μ is said to be a non-atomic measure if $\mu(\{x\}) = 0$ for any $x \in X$ (provided that $\{x\} \in \mathcal{B}$).

Let $(X, \mathcal{B})$ be a measurable space and let μ and ν be two measures on $\mathcal{B}$. We say that the measure μ is absolutely continuous with respect to the measure ν, and we write $\mu \ll \nu$ iff from $B \in \mathcal{B}$ and $\nu(B) = 0$ it follows that $\mu(B) = 0$. If at the same time $\mu \ll \nu$ and $\nu \ll \mu$, then we say that the measures μ and ν are equivalent, and we write $\mu \sim \nu$.

Let $(X, \mathcal{B}, \mu)$ be an arbitrary measure space. Denote by $\mathcal{B}_\mu$ the completion in measure μ of the σ-algebra $\mathcal{B}$, that is a σ-algebra of all subsets in X of the form $B \cup N$, where $B \in \mathcal{B}$ and $N \subset A$ for some $A \in \mathcal{B}$ such that $\mu(A) = 0$. The σ-algebra $\mathcal{B}_\mu$ is also called a σ-algebra of sets measurable with respect to the measure μ, or a σ-algebra of μ-measurable sets.

The measure μ, which is defined on $\mathcal{B}$, we may in a natural way extend to a measure on $\mathcal{B}_\mu$ putting $\mu(B \cup N) = \mu(B)$.

For example, if $X = R$ is a real line, $\mathcal{B} = \mathcal{B}(X)$ is the Borel σ-algebra on R and $\mu = m$ is the Lebesgue measure defined on $\mathcal{B}$, then $\mathcal{B}_m$ is a σ-algebra of sets measurable with respect to the Lebesgue measure (measurable in the sense of Lebesgue), and a measure m extended to $\mathcal{B}_m$ is the classical Lebesgue measure on R.

It is well known that if A is a Borel subset of R with positive Lebesgue measure, then there exists a subset B of A which is non-measurable with respect to m. In this paper we will need some extension of this fact.

Lemma 1 ([5]). *Let X be a separable complete metric space and $\mathcal{B} = \mathcal{B}(X)$ the Borel σ-algebra on X. Suppose that μ is a σ-finite and non-atomic measure on $\mathcal{B}$. If $A \in \mathcal{B}$ and $\mu(A) > 0$, then there exists a set $B \subset A$ such that $B \notin \mathcal{B}_\mu$ (i.e. B is non-measurable with respect to μ).*

Let $(X, \mathcal{B}, \mu)$ be an arbitrary measure space, and $(Y, \mathcal{F})$ a measurable space. A mapping $f \colon X \to Y$ is called a μ-measurable mapping if it is measurable with respect to $(\mathcal{B}_\mu, \mathcal{F})$, i.e. if $f^{-1}(A) \in \mathcal{B}_\mu$ for each $A \in \mathcal{F}$.

Of course, every measurable mapping $f : X \to Y$ is μ-measurable for any measure μ on $\mathcal{B}$.

2. Main results

In this section we shall deal with the generalization of the theorem of Suslin-Kuratowski to the case of μ-measurable mappings. We give conditions under which such a generalization is possible.

Throughout this section X will always denote a separable complete metric space, $\mathcal{B}$ the Borel σ-algebra on X and μ a Borel measure on X. As usual by $\mathcal{B}_\mu$ we shall denote the σ-algebra of μ-measurable sets.

Now let f be a μ-measurable mapping from X into X, i.e. for any $B \in \mathcal{B}$, $f^{-1}(B) \in \mathcal{B}_\mu$.

In order to prove the main results of this paper we will need the following statement.

Lemma 2. *For every μ-measurable mapping $f : X \to X$ there exists a Borel subset X_0 of X such that $\mu(X - X_0) = 0$ and the restriction $f|_{X_0}$ of the mapping f to the set X_0 is Borel measurable.*

Proof. It is clear that since X is a separable metric space, then there exists a countable class $\Gamma = \{K_n, n = 1, 2, \dots\}$ of Borel subsets of X such that the Borel σ-algebra $\mathcal{B}$ is generated by this class, i.e. $\mathcal{B} = \sigma(\Gamma)$ ($\sigma(\Gamma)$ denotes the least σ-algebra which contains Γ). Indeed, if E is a countable dense set in X, then as a class Γ we may take a set of all balls of radius r with centre at the point x, where r is an arbitrary rational number and x is an arbitrary element of E.

Let $B_n = f^{-1}(K_n)$, for $n = 1, 2, \dots$. Obviously $B_n \in \mathcal{B}_\mu$. Therefore $B_n = A_n \cup C_n$, where $A_n \in \mathcal{B}$, $C_n \subset D_n$, $D_n \in \mathcal{B}$ and $\mu(D_n) = 0$.

Put $D = \bigcup_{n=1}^{\infty} D_n$. Then $D \in \mathcal{B}$ and $\mu(D) = 0$. Now let $X_0 = X - D$. Then X_0 is a Borel subset of X and $\mu(X - X_0) = 0$.

Denote by g the restriction of the mapping f to the set X_0, i.e. $g = f|_{X_0}$. We must show that g is a Borel mapping. To prove this fact it is enough to show that $g^{-1}(B) \in \mathcal{B}$ for every set $B \in \Gamma$ (since $\sigma(\Gamma) = \mathcal{B}$), i.e. more precisely that for any $n = 1, 2, \dots,\ g^{-1}(K_n) \in \mathcal{B}$.

Indeed, we have that $g^{-1}(K_n) = \{x \in X_0 : g(x) \in K_n\} = \{x \in X_0 : f(x) \in K_n\} = f^{-1}(K_n) \cap X_0 = B_n \cap X_0 = (A_n \cup C_n) \cap X_0 = (A_n \cap X_0) \cup (C_n \cap X_0) = A_n \cap X_0$, since for any $n = 1, 2, \dots,\ C_n \cap X_0 = \varnothing$ (which follows from the fact that $C_n \subset D_n \subset D = X - X_0$). Therefore for any $n = 1, 2, \dots,\ g^{-1}(K_n) = A_n \cap X_0$, and since $A_n \in \mathcal{B}$ and $X_0 \in \mathcal{B}$, then $A_n \cap X_0 \in \mathcal{B}$, i.e. $g^{-1}(K_n) \in \mathcal{B}$. This completes the proof.

Now we are ready to extend the results from [5], where the generalization of the theorem of Suslin-Kuratowski for Borel mappings was considered, to the case of μ-measurable mappings.

Proposition 1. *Let f be a μ-measurable and one-to-one mapping from X into X. If $\mu \ll \mu_f$, then $f(B) \in \mathcal{B}_\mu$ for every set $B \in \mathcal{B}_\mu$.*

Proof. In view of Lemma 2 there exists a Borel subset X_0 of X such that $\mu(X - X_0) = 0$ and the restriction of f to X_0 is Borel measurable. Suppose that $\mu \ll \mu_f$ and let B be an arbitrary μ-measurable set, i.e. $B \in \mathcal{B}_\mu$.

Put $B_1 = B \cap X_0$ and $B_2 = B \cap (X - X_0)$. Then $B = B_1 \cup B_2$, whence

$$f(B) = f(B_1) \cup f(B_2). \tag{1}$$

First of all we show that $f(B_1) \in \mathcal{B}_\mu$. Since $B_1 \in \mathcal{B}_\mu$ and $B_1 \subset X_0$, then $B_1 = A \cup N$, where $A \in \mathcal{B}$ and $A \subset X_0$, $N \subset A_1$, $A_1 \in \mathcal{B}$ and $\mu(A_1) = 0$. We have therefore that $f(B_1) = f(A) \cup f(N)$. Since A is a Borel subset of X_0 and $f|_{X_0}$ is Borel measurable, then by virtue of the Theorem of Kuratowski we infer that $f(A) \in \mathcal{B}$ and consequently $f(A) \in \mathcal{B}_\mu$. Note also that $f(N) \in \mathcal{B}_\mu$. Indeed, let $A_0 = A_1 \cap X_0$. Then $A_0 \in \mathcal{B}$ and $A_0 \subset X_0$, and applying again the Theorem of Kuratowski we get that $f(A_0) \in \mathcal{B}$. Moreover, $N \subset A_1$ and $N \subset B \subset X_0$, whence $N \subset A_0$. Since f is an injection, we have that $f^{-1}(f(A_0)) = A_0$. Hence $\mu_f(f(A_0)) = \mu(f^{-1}(f(A_0))) = \mu(A_0)$. But $\mu(A_0) = 0$, since $A_0 \subset A_1$ and $\mu(A_1) = 0$. Therefore $\mu_f(f(A_0)) = 0$, whence by virtue of the assumption (i.e. $\mu \ll \mu_f$) we obtain that $\mu(f(A_0)) = 0$. From $N \subset A_0$ it follows that $f(N) \subset f(A_0)$, which implies that $f(N) \in \mathcal{B}_n$. Hence, since also $f(A) \in \mathcal{B}_\mu$ and $f(B_1) = f(A) \cup f(N)$, we get that $f(B_1) \in \mathcal{B}_\mu$.

Now we show that $f(B_2) \in \mathcal{B}_\mu$. Note in the first place that since $X_0 \in \mathcal{B}$ and $f|_{X_0}$ is Borel measurable, then in view of the Theorem of Kuratowski $f(X_0) \in \mathcal{B}$. From $X = X_0 \cup (X - X_0)$ it follows that $f(X) = f(X_0) \cup f(X - X_0)$, and since f is a one-to-one mapping and $X_0 \cap (X - X_0) = \varnothing$, then $f(X_0) \cap f(X - X_0) = \varnothing$. Since $X - f(X_0) \in \mathcal{B}$, applying again the fact that f is an injection, we have that $\mu_f(X - f(X_0)) = \mu(f^{-1}(X - f(X_0))) = \mu(f^{-1}(X) - f^{-1}(f(X_0))) = \mu(X - X_0) = 0$, i.e. $\mu_f(X - f(X_0)) = 0$. But $B_2 \subset X - X_0$, whence $f(B_2) \subset f(X - X_0) = f(X) - f(X_0) \subset X - f(X_0)$, which consequently implies that $f(B_2) \in \mathcal{B}_\mu$.

We showed therefore that $f(B_1) \in \mathcal{B}_\mu$ and $f(B_2) \in \mathcal{B}_\mu$. Hence taking into account (1) we infer that also $f(B) \in \mathcal{B}_\mu$, and that is what we wished to prove.

Remark. Proposition 1 is also true without the assumption that f is a one-to-one mapping, but only for the μ-measurable sets of full measure (and only for σ-finite measures). This follows from the following statement.

Proposition 2. *Assume that the measure μ is σ-finite, and let f be a μ-measurable mapping from X into X. If $\mu \ll \mu_f$, then for every set $B \in \mathcal{B}_\mu$ such that $\mu(X - B) = 0$ we have that $f(B) \in \mathcal{B}_\mu$ and $\mu(X - f(B)) = 0$.*

In particular, if μ is a probability [finite] measure, then from $B \in \mathcal{B}_\mu$ and $\mu(B) = 1$ [$\mu(B) = \mu(X)$] it follows that $f(B) \in \mathcal{B}_\mu$ and $\mu(f(B)) = 1$ [$\mu(f(B)) = \mu(X)$].

Proof. The proof will be divided into two steps. Suppose in the first place that the measure μ is finite. For simplicity we may obviously assume that μ is a probability measure, i.e. $\mu(X) = 1$.

By virtue of Lemma 2 there exists a Borel subset X_0 of X such that $\mu(X - X_0) = 0$, i.e. $\mu(X_0) = 1$, and the restriction $f|_{X_0}$ is Borel measurable.

Now let $B \in \mathcal{B}_\mu$ and $\mu(B) = 1$. From the definition of the σ-algebra $\mathcal{B}_\mu$ we have that there is a Borel subset B_1 of B such that $\mu(B_1) = 1$. Put $B_0 = B_1 \cap X_0$. Then $B_0 \in \mathcal{B}$ and $\mu(B_0) = 1$. Since $B_0 \subset X_0$, $X_0 \in \mathcal{B}$, $\mu(X_0) = 1$ and $f|_{X_0}$ is a Borel mapping, we infer by virtue of the Lusin theorem (see [3, Corollary 24.22]) that for any $n = 1, 2, \ldots$ there exists a compact subset K_n of B_0 such that $\mu(B_0 - K_n) < 1/n$ and the restriction $f|_{K_n}$ of f to K_n is a continuous mapping on K_n.

Put $X_1 = \bigcup_{n=1}^{\infty} K_n$ and $Y_1 = \bigcup_{n=1}^{\infty} f(K_n)$. It is clear that $X_1 \in \mathcal{B}$ and $\mu(X_1) = 1$. Moreover, since f is a continuous mapping on K_n, then $f(K_n)$ is a compact set. Therefore Y_1 is a σ-compact and consequently a Borel subset of X, i.e. $Y_1 \in \mathcal{B}$. Furthermore we have that

$$f^{-1}(Y_1) = \bigcup_{n=1}^{\infty} f^{-1}(f(K_n)) \supset \bigcup_{n=1}^{\infty} K_n = X_1.$$

Thus $f^{-1}(Y_1) \supset X_1$, whence $\mu_f(Y_1) = \mu(f^{-1}(Y_1)) \geq \mu(X_1) = 1$. Therefore $\mu_f(Y_1) = 1$, and taking into account that $\mu \ll \mu_f$ we obtain that $\mu(Y_1) = 1$. Moreover, since for any $n = 1, 2, \ldots,$ $K_n \subset B_0$, then $f(K_n) \subset f(B_0)$, and consequently $\bigcup_{n=1}^{\infty} f(K_n) \subset f(B_0)$, i.e. $Y_1 \subset f(B_0)$. Hence $f(B_0) \in \mathcal{B}_\mu$ and $\mu(f(B_0)) = 1$. But

$B_0 \subset B_1 \subset B$, whence $f(B_0) \subset f(B)$. Therefore $f(B) \in \mathcal{B}_\mu$ and $\mu(f(B)) = 1$, which completes the first part of the proof, for the case when the measure μ is finite.

Suppose now that μ is an arbitrary σ-finite measure. This means that $X = \bigcup_{n=1}^{\infty} X_n$, where for any $n = 1, 2, \dots,$ $X_n \in \mathcal{B}$ and $\mu(X_n) < \infty$. Let $B \in \mathcal{B}_\mu$ and $\mu(X - B) = 0$. If we set $B_n = B \cap X_n$ (for $n = 1, 2, \dots$), then $B_n \in \mathcal{B}_n$ and the fact that $X_n - B_n \subset X - B$ gives that $\mu(X_n - B_n) = 0$. Since $B_n \subset X_n$ and $\mu(X_n) < \infty$, then identical considerations as in the first part of this proof show that $f(B_n) \in \mathcal{B}_\mu$ and $\mu(f(B_n)) = \mu(X_n)$, i.e. $\mu(X_n - f(B_n)) = 0$. But $B = \bigcup_{n=1}^{\infty} B_n$, whence $f(B) = \bigcup_{n=1}^{\infty} f(B_n)$. Therefore also $f(B) \in \mathcal{B}_\mu$. Moreover $X - f(B) = \bigcup_{n=1}^{\infty} X_n - \bigcup_{n=1}^{\infty} f(B_n) \subset \bigcup_{n=1}^{\infty}(X_n - f(B_n))$. Hence $\mu(X - f(B)) \leq \sum_{n=1}^{\infty} \mu(X_n - f(B_n)) = 0$, i.e. $\mu(X - f(B)) = 0$. The proposition is thus proved.

There is a question if the theorem inverse to Proposition 1 is also true. Now we show that such an inverse theorem is in reality true, if we make some additional assumptions about the measure μ and the mapping f. However, as opposed to Proposition 1 we do not assume that f is an injection.

Proposition 3. *Suppose that the measure μ is σ-finite and non-atomic, and f is a μ-mesurable mapping from X onto X (i.e. $f(X) = X$). If $f(B) \in \mathcal{B}_\mu$ for every set $B \in \mathcal{B}_\mu$, then $\mu \ll \mu_f$.*

Proof. Suppose that our assertion is not true. This means that there exists a set $B_1 \in \mathcal{B}$ such that $\mu_f(B_1) = 0$ and $\mu(B_1) > 0$. Then $f^{-1}(B_1) \in \mathcal{B}_\mu$ and from the definition of μ_f we have that $\mu(f^{-1}(B_1)) = 0$. Since $B_1 \in \mathcal{B}$, $\mu(B_1) > 0$ and μ is a σ-finite and non-atomic measure, we get from Lemma 1 that there is a set $A_1 \subset B_1$ which is not μ-measurable (i.e. $A_1 \notin \mathcal{B}_\mu$). Let $B = f^{-1}(A_1)$. Evidently $B \neq \varnothing$, which follows from the fact that $A_1 \subset f(X) = X$. Since $A_1 \subset B_1$, then $f^{-1}(A_1) \subset f^{-1}(B_1)$, i.e. $B \subset f^{-1}(B_1)$. But $f^{-1}(B_1) \in \mathcal{B}_\mu$ and $\mu(f^{-1}(B_1)) = 0$, which implies that $B \in \mathcal{B}_\mu$. On the other hand we have however that $f(B) = f(f^{-1}(A_1)) = A_1$ (since $f(X) = X$). But $A_1 \notin \mathcal{B}_\mu$, that is $f(B) \notin \mathcal{B}_\mu$. Therefore we have found a set $B \in \mathcal{B}_\mu$ such that $f(B) \notin \mathcal{B}_\mu$. But this contradicts the assumption and consequently completes the proof.

Remark. Proposition 3 is not true if we do not assume that f is an onto mapping (even if we suppose that f is an injection). This follows from the following example.

Let $X = [0, 1]$ be a unit interval, $\mathcal{B}$ the Borel σ-algebra on X and $\mu = m$ the Lebesgue measure on $\mathcal{B}$. We define a mapping $f : [0, 1] \to [0, 1]$ putting $f(x) = \frac{1}{2}x$, for $x \in [0, 1]$. Obviously f is a one-to-one Borel mapping but not a surjection (since $f([0, 1]) = [0, \frac{1}{2}]$). It is easy to check that $f(B) \in \mathcal{B}_m$ for every set $B \in \mathcal{B}_m$ (see [3], Th. 21.1), but it is not true that $\mu \ll \mu_f$. Indeed, if we put for example $B = (\frac{1}{2}, 1]$, then $\mu_f(B) = \mu(f^{-1}(B)) = \mu(\varnothing) = 0$, but on the other hand $\mu(B) = \frac{1}{2}$. Therefore the assumption in Proposition 3 that f is a surjection is in reality essential.

As a corollary from Propositions 1 and 3 we thus obtain the following theorem.

Theorem 1. *Let X be a separable complete metric space and μ a σ-finite non-atomic measure defined on the Borel σ-algebra $\mathcal{B}$ of the space X. Assume that f is a one-to-one μ-measurable mapping from X onto X. Then the following two conditions are equivalent*:

(a) *$f(B) \in \mathcal{B}_\mu$ for every set $B \in \mathcal{B}_\mu$.*

(b) *$\mu \ll \mu_f$.*

At the end of this paper we shall prove some other properties of μ-measurable mappings on a metric space X, similar to Propositions 1 and 3, but concerning inverse images of μ-measurable sets. It is possible to treat the facts that follow (Propositions 4 and 5) as a peculiar form of the Theorem of Kuratowski for inverse images.

Recall that by definition a mapping $f : X \to X$ is μ-measurable if

$$f^{-1}(B) \in \mathcal{B}_\mu \quad \text{for any set } B \in \mathcal{B}. \tag{2}$$

There arises the question if in this definition we can exchange the condition $B \in \mathcal{B}$ by the condition $B \in \mathcal{B}_\mu$. In the other words there is the problem if for a μ-measurable mapping $f : X \to X$ the property

$$f^{-1}(B) \in \mathcal{B}_\mu \quad \text{for any set } B \in \mathcal{B}_\mu \tag{3}$$

is true.

Unfortunately, this fact need not be true, even if f is a Borel mapping. Indeed, in Example 2 of [5] we have shown, that if ν is a Gaussian probability measure on R and $\mu = \nu \times \nu \times \dots$ is the product measure on the linear metric space $X = R^\infty = R \times R \times \cdots$, then the μ-completion $\mathcal{B}_\mu$ of the Borel σ-algebra $\mathcal{B} = \mathcal{B}(R^\infty)$ is not invariant under all translations. This means, that there exists a set $B \in \mathcal{B}_\mu$ and an element $x_0 \in R^\infty$ such that $B + x_0 \notin \mathcal{B}_\mu$. Now let $f : R^\infty \to R^\infty$ be a translation on R^∞ given by the formula $f(x) = x - x_0$ for $x \in R^\infty$. Clearly, f is a Borel mapping on R^∞. Moreover, $f^{-1}(B) = \{x \in R^\infty : f(x) \in B\} = \{x \in R^\infty : x - x_0 \in B\} = \{x \in R^\infty : x \in B + x_0\} = B + x_0$. Therefore $f^{-1}(B) \notin \mathcal{B}_\mu$.

Nevertheless, it is easy to prove that the condition (3) will be true for a μ-measurable mapping f if we assume that the measure μ_f is absolutely continuous with respect to μ (i.e. that $\mu_f \ll \mu$).

We receive therefore the following assertion which is similar to Proposition 1. But as opposed to this proposition we don't need to assume that f is an injection.

Proposition 4. *Let f be a μ-measurable mapping from X into X. If $\mu_f \ll \mu$, then $f^{-1}(B) \in \mathcal{B}_\mu$ for every set $B \in \mathcal{B}_\mu$.*

Proof. Thus let $B \in \mathcal{B}_\mu$, i.e. $B = A \cup N$ where $A \in \mathcal{B}$, $N \subset A_1$, $A_1 \in \mathcal{B}$ and $\mu(A_1) = 0$. Then

$$f^{-1}(B) = f^{-1}(A) \cup f^{-1}(N). \tag{4}$$

Since $A \in \mathcal{B}$ and $A_1 \in \mathcal{B}$, then $f^{-1}(A) \in \mathcal{B}_\mu$ and $f^{-1}(A_1) \in \mathcal{B}_\mu$. Moreover $\mu(f^{-1}(A_1)) = \mu_f(A_1)$. But $A_1 \in \mathcal{B}$ and $\mu(A_1) = 0$. Hence, since $\mu_f \ll \mu$, we obtain that $\mu_f(A_1) = 0$, and consequently $\mu(f^{-1}(A_1)) = 0$. But $f^{-1}(N) \subset f^{-1}(A_1)$, which implies that $f^{-1}(N) \in \mathcal{B}_\mu$. Since also $f^{-1}(A) \in \mathcal{B}_\mu$, then from (4) we conclude that $f^{-1}(B) \in \mathcal{B}_\mu$, and that is what we had to prove.

The theorem inverse to the above proposition is also true, if we assume that the measure μ is σ-finite and non-atomic and f is an injection. Thus we have the following fact, analogous to Proposition 3.

Proposition 5. *Suppose that the measure μ is σ-finite and non-atomic and f is a one-to-one μ-measurable mapping from X into X. If $f^{-1}(B) \in \mathcal{B}_\mu$ for every set $B \in \mathcal{B}_\mu$, then $\mu_f \ll \mu$.*

Proof. Assume that our assertion is not true. Then there exists a Borel subset B_1 such that $\mu(B_1) = 0$ and $\mu_f(B_1) > 0$.

In view of Lemma 2 the restriction $f|_{X_0}$ is Borel measurable for some Borel set X_0 such that $\mu(X - X_0) = 0$. Put $B_0 = B_1 \cap f(X_0)$. Then $B_0 \subset f(X_0) \subset f(X)$. Moreover, from the Theorem of Kuratowski it follows that $f(X_0) \in \mathcal{B}$. Therefore also $B_0 \in \mathcal{B}$, and since $B_0 \subset B_1$, then $\mu(B_0) = 0$.

Observe now that $\mu_f(X-f(X_0))=\mu(f^{-1}(X-f(X_0)))=\mu(f^{-1}(X)-f^{-1}(f(X_0)))$ $= \mu(X - X_0) = 0$, i.e. $\mu_f(X - f(X_0)) = 0$.

Furthermore, from $B_0 \cup (B_1 - B_0) = B_1$ we have that $\mu_f(B_0) + \mu_f(B_1 - B_0) = \mu_f(B_1)$. Hence taking into account that $\mu_f(B_1 - B_0) = 0$ (since $B_1 - B_0 \subset X - f(X_0)$), we conclude that $\mu_f(B_0) = \mu_f(B_1)$, whence in particular $\mu_f(B_0) > 0$.

We have shown therefore that there exists a Borel subset B_0 of X such that $B_0 \subset f(X)$, $\mu(B_0) = 0$ and $\mu_f(B_0) > 0$, i.e. $\mu(f^{-1}(B_0)) > 0$.

Let $A = f^{-1}(B_0)$. Then $A \in \mathcal{B}_\mu$ and $\mu(A) > 0$. In view of the definition of the σ-algebra $\mathcal{B}_\mu$ this means that there exists a set $A_0 \in \mathcal{B}$ such that $A_0 \subset A$ and $\mu(A_0) = \mu(A) > 0$. Hence by virtue of Lemma 1 we obtain that there is a set $A_1 \subset A_0$ (and consequently $A_1 \subset A$) such that $A_1 \notin \mathcal{B}_\mu$.

Now let $B = f(A_1)$. Since $A_1 \subset A$, then $f(A_1) \subset f(A)$. But $f(A) = f(f^{-1}(B_0))$ $= B_0$ (since $B_0 \subset f(X)$). Hence $B \subset B_0$ and since $B_0 \in \mathcal{B}$ and $\mu(B_0) = 0$, then $B \in \mathcal{B}_\mu$. Therefore by virtue of the assumption we have that also $f^{-1}(B) \in \mathcal{B}_\mu$. But on the other hand $f^{-1}(B) = f^{-1}(f(A_1)) = A_1$ (since f is an injection) and $A_1 \notin \mathcal{B}_\mu$. Thus we get a contradiction, which finishes our proof.

Remark. Proposition 5 is false if f is not an injection. Indeed, let for example $X = [0, 1]$, $\mathcal{B}$ be the Borel σ-algebra on X and $\mu = m$ the Lebesgue measure on $\mathcal{B}$. If we put $f(x) = 1$ for $x \in [0, 1]$, then the assumptions of Proposition 4 are fulfilled. In fact, f is obviously a μ-measurable mapping (even Borel), and for any set $B \in \mathcal{B}_\mu$ we have that either $f^{-1}(B) = \varnothing$ (if $1 \notin B$) or $f^{-1}(B) = [0, 1]$ (if $1 \in B$). Therefore $f^{-1}(B) \in \mathcal{B}_\mu$ always. But the condition $\mu_f \ll \mu$ is not satisfied, since for example $\mu(\{1\}) = 0$ and $\mu_f(\{1\}) = 1 > 0$ (μ_f is a probability measure concentrated at the point 1).

As a corollary from Propositions 4 and 5 we obtain the following theorem (similar to Theorem 1, except for the assumption that f is a surjection).

Theorem 2. *Let X be a separable complete metric space and μ a σ-finite non-atomic measure defined on the Borel σ-algebra $\mathcal{B}$ of the space X. Assume that f is a one-to-one μ-measurable mapping from X into X. Then the following two conditions are equivalent*:

(a) $f^{-1}(B) \in \mathcal{B}_\mu$ *for every set* $B \in \mathcal{B}_\mu$.

(b) $\mu_f \ll \mu$.

We finish the paper with the following corollary from our results. Namely, combining Propositions 1 and 3–5 we get the theorem which gives necessary and sufficient conditions in order that for every set $B \in \mathcal{B}_\mu$, $f(B) \in \mathcal{B}_\mu$ and $f^{-1}(B) \in \mathcal{B}_\mu$ for a μ-measurable mapping f on X.

Theorem 3. *Let X be a separable complete metric space and μ a measure defined on the Borel σ-algebra $\mathcal{B}$ of the space X. Assume that f is a one-to-one and μ-measurable mapping from X into X. In order that $f(B) \in \mathcal{B}_\mu$ and $f^{-1}(B) \in \mathcal{B}_\mu$ for every set $B \in \mathcal{B}_\mu$ it is sufficient and, if μ is a non-atomic and σ-finite measure and f is a surjection, it is also necessary that the measures μ_f and μ are equivalent (i.e. $\mu_f \sim \mu$).*

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Institute of Mathematics, Szczecin University, ul. Wielkopolska 15, 70-451 Szczecin, Poland

E-mail address: awisniew@uoo.univ.szczecin.pl