

THE CENTRAL INTERTWINING LIFTING AND STRICT CONTRACTIONS

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ABSTRACT. In this paper we give a necessary and sufficient condition for the central intertwining lifting of a strict contraction to be strictly contractive. As an application, we obtain a factorization of $D_{A_c}^{-2}$ when the central intertwining lifting A_c of A is a strict contraction.

Let \mathcal{H} , \mathcal{H}' be (complex, separable) Hilbert spaces, T and T' contractions in $\mathcal{L}(\mathcal{H})$ and $\mathcal{L}(\mathcal{H}')$, respectively, having minimal isometric dilations U and U' in $\mathcal{L}(\mathcal{K})$ and $\mathcal{L}(\mathcal{K}')$. If A is a contraction operator in $\mathcal{L}(\mathcal{H}, \mathcal{H}')$ such that $AT = T'A$, the celebrated Sz.-Nagy–Foias commutant lifting theorem ([3, 7, 8]) ensures the existence of an intertwining lifting of A , i.e. the existence of a contraction B in $\mathcal{L}(\mathcal{K}, \mathcal{K}')$ such that $BU = U'B$ and $B^*|_{\mathcal{H}'} = A^*$. Recently, a particular intertwining lifting of A , called the *central intertwining lifting* of A (see [4, 5, 6]) has played an important role in solving certain \mathbf{H}^2 - \mathbf{H}^∞ optimization problems appearing in control theory. Moreover, an example of a strictly contractive Hankel operator whose central intertwining lifting has norm one was given in [2]. The purpose of this note is to prove a necessary and sufficient condition (Theorem 1 below) for the central intertwining lifting A_c of a strict contraction A to be strictly contractive. Necessary and sufficient conditions for the central intertwining lifting to be strictly contractive were also obtained in [5] when T is a unilateral shift. In this case, the central intertwining lifting is strictly contractive if and only if the outer function $(I - A^*A)|_{\ker T^*}^{-1}$ is invertible (see [5]).

For the convenience of the reader we begin by reviewing some notation and terminology (see [3, 4, 5, 6, 8]). Throughout this note all Hilbert spaces are understood to be complex and separable. If \mathcal{D} , \mathcal{D}' are Hilbert spaces and $C \in \mathcal{L}(\mathcal{D}, \mathcal{D}')$ is a contraction (i.e. $\|C\| \leq 1$), we denote as usual the defect operator of C by D_C ($:= (I - C^*C)^{1/2}$), the defect space of C by \mathcal{D}_C ($:= \overline{D_C \mathcal{D}}$), and C is said to be a strict contraction if $\|C\| < 1$. Let \mathbf{D} denote the open unit disc in \mathbf{C} , and $\mathbf{T} = \partial \mathbf{D}$. The spaces $\mathbf{L}^2(\mathcal{D})$ and $\mathbf{H}^2(\mathcal{D})$ are the usual Lebesgue and Hardy spaces of \mathcal{D} -valued functions on the unit circle \mathbf{T} relative to normalized Lebesgue measure \mathbf{m} on \mathbf{T} , and $S \in \mathcal{L}(\mathbf{H}^2(\mathcal{D}))$ is the multiplication by the variable on $\mathbf{H}^2(\mathcal{D})$. Since the minimal isometric dilation of T' is unique (up to an isomorphism (see [3, 8])), we may and do assume that $U' \in \mathcal{L}(\mathcal{K}')$, the minimal isometric dilation of T' is the Sz.-Nagy–Schäffer minimal isometric dilation of T' , i.e. $\mathcal{K}' = \mathcal{H}' \oplus \mathbf{H}^2(\mathcal{D}_{T'})$ and U'

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on $\mathcal{H}' \oplus \mathbf{H}^2(\mathcal{D}_{T'})$ is given by

$$U' = \begin{bmatrix} T' & 0 \\ D_{T'} & S' \end{bmatrix} : \begin{matrix} \mathcal{H}' \\ \oplus \\ \mathbf{H}^2(\mathcal{D}_{T'}) \end{matrix} \rightarrow \begin{matrix} \mathcal{H}' \\ \oplus \\ \mathbf{H}^2(\mathcal{D}_{T'}) \end{matrix},$$

where $S' \in \mathcal{L}(\mathbf{H}^2(\mathcal{D}_{T'}))$ is the multiplication by z on $\mathbf{H}^2(\mathcal{D}_{T'})$, (see [3, 8]). If A is a strict contraction, then the central intertwining lifting A_c of A is the contraction in $\mathcal{L}(\mathcal{K}, \mathcal{H}' \oplus \mathbf{H}^2(\mathcal{D}_{T'}))$ given by

$$(1) \quad A_c = \begin{bmatrix} \tilde{A} \\ D_{T'} \tilde{A} T_{\tilde{A}}^* (I - z T_{\tilde{A}}^*)^{-1} \end{bmatrix} : \mathcal{K} \rightarrow \begin{matrix} \mathcal{H}' \\ \oplus \\ \mathbf{H}^2(\mathcal{D}_{T'}) \end{matrix},$$

where $\tilde{A} := AP$, $T_{\tilde{A}} := (I - \tilde{A}^* \tilde{A})U(I - U^* \tilde{A}^* \tilde{A}U)^{-1}$, and P is the orthogonal projection from \mathcal{K} onto \mathcal{H} (see [4, 5]). Moreover, let $N_* \in \mathcal{L}(\mathcal{D}_{T'})$, $L \in \mathcal{L}(\mathcal{H})$ be the operators defined by

$$(2) \quad \begin{aligned} N_* &= I_{\mathcal{D}_{T'}} + D_{T'} A D_A^{-2} A^* D_{T'}, \\ L &= D_A^{-2} - (I - T^* A^* A T)^{-1}. \end{aligned}$$

Moreover, for any $k \in \mathcal{K}$ the function

$$(3) \quad z \rightarrow k_A(z) := D_{T'} A D_A^{-2} P (I - z U^*)^{-1} k$$

on \mathbf{D} will be denoted by k_A . Now we may state the main result of this note.

Theorem 1. *Let T and T' be contractions in $\mathcal{L}(\mathcal{H})$ and $\mathcal{L}(\mathcal{H}')$, respectively, having minimal isometric dilations $U \in \mathcal{L}(\mathcal{K})$ and $U' \in \mathcal{L}(\mathcal{H}' \oplus \mathbf{H}^2(\mathcal{D}_{T'}))$. If A is a strict contraction in $\mathcal{L}(\mathcal{H}, \mathcal{H}')$ such that $AT = T'A$, then the central intertwining lifting A_c of A is a strict contraction if and only if for any k in \mathcal{K} the function k_A is in $\mathbf{H}^2(\mathcal{D}_{T'})$. Moreover, if T is the multiplication by the variable on $\mathbf{H}^2(\mathcal{D})$, then the central intertwining lifting A_c of A is a strict contraction if and only if the function $\Theta_A: \mathbf{D} \rightarrow \mathcal{L}(\mathcal{D}_{T'}, \mathcal{D})$ given by*

$$\Theta_A(z) d_{T'} = (D_A^{-2} A^* D_{T'} d_{T'})(z), \quad z \in \mathbf{D}, d_{T'} \in \mathcal{D}_{T'}$$

is bounded.

Proof. Without loss of generality we may assume (see [3, 8]) that T is an isometry. In this case $U = T$, $P = I_{\mathcal{H}}$, and for any nonnegative integer n let $i_n : \mathcal{D}_{T'} \rightarrow \mathcal{H}' \oplus \mathbf{H}^2(\mathcal{D}_{T'})$ be defined by $i_n d_{T'} = 0 \oplus e^{int} d_{T'}$, $d_{T'} \in \mathcal{D}_{T'}$, and let $\mathcal{H}'_n := \mathcal{H}' \oplus \bigoplus_{k=0}^n e^{ikt} \mathcal{D}_{T'} (\subset \mathcal{H}' \oplus \mathbf{H}^2(\mathcal{D}_{T'}))$. Furthermore, let P'_n denote the orthogonal projection from $\mathcal{H}' \oplus \mathbf{H}^2(\mathcal{D}_{T'})$ onto \mathcal{H}'_n , $A_n := P'_n A_c$ and $T'_n := P'_n U' |_{\mathcal{H}'_n}$. Then (see [3, 8]), for any nonnegative integer n

$$(4) \quad \begin{aligned} P'_n &= U'^* P'_{n-1} U' + i_{n-1} i_{n-1}^*, \\ A_n T &= T'_n A_n, \end{aligned}$$

and according to (1) and Lemma V.1.1 of [3] (A_{n+1} is a one-step intertwining lifting of A_n), for any nonnegative integer n

$$(5) \quad i_n^* A_n = i_{n-1}^* A_{n-1} T_{A_{n-1}}^*.$$

We will show that for any nonnegative integer n

$$(6) \quad D_{A_n}^{-2} = D_A^{-2} + \sum_{k=1}^{n+1} T^k L T^{*k}.$$

Once we show this the proof can be completed as follows. It is easy to see that

$$(7) \quad L = D_A^{-2} A^* D_{T'} N_*^{-1} D_{T'} A D_A^{-2},$$

so by (6) and (7), $\|A_c\| < 1$ if and only if there exists a positive constant M such that for any h in \mathcal{H} ($= \mathcal{K}$)

$$(8) \quad \sum_{n=0}^{\infty} \|L^{1/2} T^{*n} h\|^2 = \sum_{n=0}^{\infty} \|N_*^{-1/2} D_{T'} A D_A^{-2} T^{*n} h\|^2 \leq M \|h\|^2.$$

Taking into account (3), it follows that for any vector h in \mathcal{H}

$$(9) \quad \|h_A\|_{\mathbf{H}^2(\mathcal{D}_{T'})}^2 = \sum_{n=0}^{\infty} \|D_{T'} A D_A^{-2} T^{*n} h\|^2,$$

so by (8) and (9) it follows that $\|A_c\| < 1$ if and only if for any vector h in \mathcal{H} , h_A is in $\mathbf{H}^2(\mathcal{D}_{T'})$ and the proof of the first part of the theorem is complete. Assume now that T is the multiplication by the variable on $\mathbf{H}^2(\mathcal{D})$ and $\|A_c\| < 1$. Then the operator $M_A : \mathbf{H}^2(\mathcal{D}) \rightarrow \mathbf{H}^2(\mathcal{D}_{T'})$ defined by $M_A := h_A$, $h \in \mathbf{H}^2(\mathcal{D})$ is bounded. It is easy to check that M_A^* is an analytic Toeplitz operator whose symbol is Θ_A (i.e. $M_A^* h' = \Theta_A h'$, $h' \in \mathbf{H}^2(\mathcal{D}_{T'})$), so Θ_A is bounded. Conversely, if Θ_A is bounded, then M_A is bounded, hence for any function h in $\mathbf{H}^2(\mathcal{D})$, $h_A \in \mathbf{H}^2(\mathcal{D}_{T'})$ and $\|A_c\| < 1$.

Now we show that (6) holds for any nonnegative integer n . From (4) and (5) we obtain that for any nonnegative integer n

$$(10) \quad \begin{aligned} D_{A_n}^2 &= D_{A_{n-1}}^2 - T_{A_{n-1}} A_{n-1}^* D_{T'}^2 A_{n-1} T_{A_{n-1}}^*, \\ D_{A_{n-1}}^2 &= I - T^* A_n^* A_n T, \end{aligned}$$

and we will show that

$$(11) \quad D_{A_n}^{-2} = D_{A_{n-1}}^{-2} + T[D_{A_{n-1}}^{-2} - D_{A_{n-2}}^{-2}]T^*,$$

from which (6) follows at once. In order to prove (11) we need to check that

$$\begin{aligned} D_{A_{n-1}}^2 T[D_{A_{n-1}}^{-2} - D_{A_{n-2}}^{-2}]T^* - T_{A_{n-1}} A_{n-1}^* D_{T'}^2 A_{n-1} T_{A_{n-1}}^* D_{A_{n-1}}^{-2} \\ - T_{A_{n-1}} A_{n-1}^* D_{T'}^2 A_{n-1} [D_{A_{n-1}}^{-2} - D_{A_{n-2}}^{-2}]T^* = 0, \end{aligned}$$

or equivalently, using (1), (4) and (10),

$$\begin{aligned} D_{A_{n-1}}^{-2} - D_{A_{n-2}}^{-2} - D_{A_{n-2}}^{-2} A_{n-1}^* D_{T'}^2 D_{A_{n-1}}^{-2} \\ - D_{A_{n-1}}^{-2} A_{n-1}^* D_{T'}^2 A_{n-1} [D_{A_{n-1}}^{-2} - D_{A_{n-2}}^{-2}] \\ = D_{A_{n-1}}^{-2} - D_{A_{n-2}}^{-2} - D_{A_{n-1}}^{-2} A_{n-1}^* D_{T'}^2 A_{n-1} D_{A_{n-1}}^{-2} \\ = D_{A_{n-1}}^{-2} [D_{A_{n-1}}^2 - D_{A_{n-2}}^2 - A_{n-1}^* D_{T'}^2 A_{n-1}] D_{A_{n-1}}^{-2} = 0, \end{aligned}$$

which follows easily since by (4) and (10)

$$\begin{aligned} D_{A_{n-1}}^2 - D_{A_{n-2}}^2 - A_{n-1}^* D_{T'_{n-1}}^2 A_{n-1} \\ = (I - T^* A_{n-1}^* A_{n-1} T) - D_{A_{n-2}}^2 - A_{n-1}^* D_{T'_{n-1}}^2 A_{n-1} \\ = (I - A_{n-1}^* T_{n-1}' T_{n-1}' A_{n-1}) - D_{A_{n-2}}^2 - A_{n-1}^* D_{T'_{n-1}}^2 A_{n-1} = 0. \end{aligned}$$

So we have proved (6) and the proof of the theorem is complete. □

A direct application of (6) yields the following.

Corollary 2. *Let T and T' be contractions in $\mathcal{L}(\mathcal{H})$ and $\mathcal{L}(\mathcal{H}')$, respectively, having minimal isometric dilations U in $\mathcal{L}(\mathcal{K})$ and U' in $\mathcal{L}(\mathcal{H}' \oplus \mathbf{H}^2(\mathcal{D}_{T'}))$. Let A in $\mathcal{L}(\mathcal{H}, \mathcal{H}')$ be a strict contraction such that $AT = T'A$, and set $\tilde{A} = AP$ (P is the orthogonal projection from \mathcal{K} onto \mathcal{H}). If the central intertwining lifting A_c of A is a strict contraction, then*

$$D_{A_c}^{-2} = \Psi^* \Psi,$$

where $\Psi : \mathcal{K} \rightarrow \mathcal{K} \oplus \mathbf{H}^2(\mathcal{D}_{T'})$ is given by

$$\Psi := \begin{bmatrix} D_{\tilde{A}}^{-2} \\ N_*^{-1/2} D_{T'} \tilde{A} D_{\tilde{A}}^{-2} (I - zU^*)^{-1} U^* \end{bmatrix}.$$

Corollary 3. *Let $h \in \mathbf{H}^\infty$ be an outer function such that $\text{dist}(\bar{h}/h, \mathbf{H}^\infty) < 1$ and $h^{-1} \notin \mathbf{H}^\infty$. If $H \in \mathcal{L}(\mathbf{H}^2, \mathbf{H}_-^2)$ is the Hankel operator defined by*

$$Hf = P_- h / \bar{h} f, \quad f \in \mathbf{H}^2,$$

then $\|H\| < 1$ and $\|H_c\| = 1$.

Proof. Without loss of generality we may assume that $h(0) = 1$. According to Chapter VIII of [9], $\|H\| < 1$ and $h^{-1} \in \mathbf{H}^2$, so the Toeplitz operator $T_{h/\bar{h}}$ is invertible. Upon setting $g := P_+ |h|^2$ we will show that

$$(12) \quad \Theta_H = D_H^{-2} H^* e^{-it} = -h^{-1} S^* g \notin \mathbf{H}^\infty$$

which by Theorem 1 will finish the proof. To show (12) let us note that since $D_H^2 = T_{h/\bar{h}}^* T_{h/\bar{h}}$ and $T_{h/\bar{h}}^* (-S^* h) = H^* e^{-it}$, it follows that

$$T_{h/\bar{h}} \Theta_H = -S^* h.$$

Thus there exists a function $u \in \mathbf{H}^2$ such that

$$(h/\bar{h}) \Theta_H = -S^* h + e^{-it} \bar{u},$$

hence

$$h \Theta_H = -\bar{h} e^{-it} (h - 1) + e^{-it} \bar{h} \bar{u} = -e^{-it} |h|^2 + e^{-it} \bar{h} (1 + \bar{u}).$$

Therefore,

$$h \Theta_H = -P_+ e^{-it} |h|^2 = -S^* P_+ |h|^2 = -S^* g.$$

Since $|h|^2 \geq 0$ a.e. on \mathbf{T} , it follows that

$$|h|^2 = SS^* g + g(0) + \overline{SS^* g}.$$

So

$$\bar{h} = h^{-1} SS^* g + g(0) h^{-1} + h^{-1} \overline{SS^* g},$$

from which it follows that

$$h^{-1} = g(0)^{-1}(\bar{h} - h^{-1}SS^*g - h^{-1}\overline{SS^*g}).$$

Since $h^{-1} \notin \mathbf{H}^\infty$, $h^{-1}SS^*g$ cannot be bounded. So $\Theta_H \notin \mathbf{H}^\infty$, and the proof of the corollary is complete. \square

Remark. The example in [2] was obtained by considering the function $h(z) = (1 - z)^\beta$, where $0 < \beta < 1/2$. Then $h \in \mathbf{H}^\infty$, $\text{dist}(\bar{h}/h, \mathbf{H}^\infty) < 1$ (see [1]), and $h^{-1} \notin \mathbf{H}^\infty$. So by Corollary 3 the central intertwining lifting of $H_{h/\bar{h}}$ has norm one and one regains the main result in [2].

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