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AN INEQUALITY FOR POLYHEDRA AND IDEAL TRIANGULATIONS OF CUSPED HYPERBOLIC 3-MANIFOLDS

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Abstract. It is not known whether every noncompact hyperbolic 3-manifold of finite volume admits a decomposition into ideal tetrahedra. We give a partial solution to this problem: Let M be a hyperbolic 3-manifold obtained by identifying the faces of n convex ideal polyhedra $P_1, \ldots, P_n$. If the faces of $P_1, \ldots, P_{n-1}$ are glued to P_n, then M can be decomposed into ideal tetrahedra by subdividing the P_i's.

1. Introduction

The famous lecture notes on 3-manifolds by Thurston [3] start with a decomposition of the figure eight knot complement into two ideal hyperbolic tetrahedra. Such a decomposition gives a convenient standpoint for the study of hyperbolic 3-manifolds, such as calculating deformations of the hyperbolic structure and computation of geometrical invariants [2]. It is thus natural to ask:

Question. *Let M be a noncompact hyperbolic 3-manifold of finite volume. Can we decompose M into ideal tetrahedra?*

Although Epstein and Penner have shown that such a manifold always admits a decomposition into convex ideal polyhedra [1], the answer to this question is not yet known. In [4], one of the authors has given an affirmative answer to the above question in the case where M is obtained by gluing two convex ideal polyhedra in a certain way. In this paper, we generalize this result to the case where M is composed of any number of convex ideal polyhedra:

Main Theorem. *Let M be a non-compact complete hyperbolic 3-manifold of finite volume which is obtained from n convex ideal polyhedra $P_1, \ldots, P_n$ by identifying the faces in pairs. Suppose that each face of P_i $(i = 1, \ldots, n-1)$ is pasted with a face of P_n, and the possibly remaining faces of P_n glued in pairs. Then M can be decomposed into non-degenerate ideal tetrahedra by subdividing the P_i's.*

The key to our result is an inequality for polyhedra. Besides Euler's theorem and a few obvious equalities, not much seems to be known about the relations among the numbers of vertices, edges, and faces of a polyhedron; we feel that our inequality

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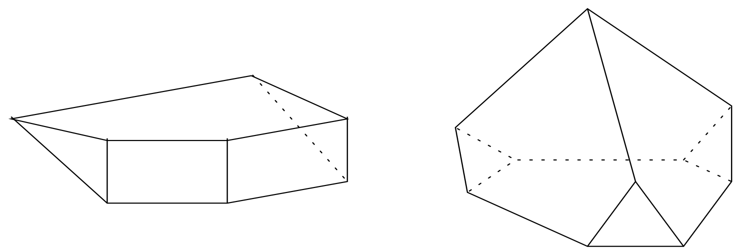

FIGURE 1

is of interest in its own right. We first introduce this inequality, and then prove the Main Theorem in section 3.

2. An inequality for polyhedra

Let P be a polyhedron, and $V = V(P)$, $E = E(P)$, $F = F(P)$, and $F_d = F_d(P)$ denote respectively the sets of vertices, edges, faces, and d-gonal faces of P.

Theorem 1. *In the notation above, we have*

$$|V|(|V|-1) \geq 8|F_4| + \sum_{d\geq 5} d(d-1)|F_d|. \tag{1}$$

Equality in the above holds if and only if P is combinatorially equivalent to one of the polyhedra depicted in Figure 1.

There is a slightly different way to view the inequality. A line segment connecting two vertices of P is called a *diagonal* of P if it goes through the interior of P; the set of diagonals of P is denoted by $\Delta(P)$. Among the $\frac{1}{2}|V|(|V|-1)$ line segments connecting two vertices of P, the number of those which lie entirely in the boundary of P is $\sum_{d\geq 3} \frac{1}{2} d(d-2)|F_d|$. Therefore Theorem 1 may be restated as:

Theorem 2. *The number of diagonals of a polyhedron P satisfies*

$$|\Delta(P)| \geq -\frac{3}{2}|F_3(P)| + \sum_{d\geq 5} \frac{d}{2}|F_d(P)|.$$

Equality in the above holds if and only if P is combinatorially equivalent to one of the polyhedra shown in Figure 1.

For our purposes, it is convenient to introduce the following notion.

Definition. For a polygon f we define $D(f)$ by

$$D(f) = \begin{cases} 0 & \text{if } \deg f = 3, \\ 2 & \text{if } \deg f = 4, \\ d-1 & \text{if } \deg f = d \geq 5. \end{cases}$$

The *capacity* of f is defined to be

$$C(f) = D(f) \deg f.$$

It depends only on the degree of f; we also write $C(d)$ for the capacity of a d-gon. Finally the capacity of a sequence $(d_1, \ldots, d_r)$ of integers with $d_i \geq 3$ is defined to be the sum $\sum_{i=1}^{r} C(d_i)$.

In terms of capacity, the inequality (1) is written as

$$|V|(|V|-1) \geq \sum_{f\in F} C(f). \tag{2}$$

Proof of Theorem 1. Suppose that $|V| = p$, $|E| = q$, $|F| = r$, and that the faces of P are of degree $d_1, \dots, d_r$. We may assume that

$$r-1 \geq d_1 \geq \cdots \geq d_r \geq 3. \tag{3}$$

Since $q = \frac{1}{2}\sum_{i=1}^{r} d_i$, Euler's theorem implies that

$$p = 2 + \frac{1}{2}\sum_{i=1}^{r} d_i - r. \tag{4}$$

We divide the proof of Theorem 1 into two cases. First suppose that $r \geq 9$. In this case the inequality (2) is a direct consequence of the following lemma:

Lemma. *Let* $(d_1, \dots, d_r)$ *be a sequence of integers satisfying* (3). *If* $r \geq 9$ *then for* p *defined by* (4) *we have*

$$p(p-1) > \sum_{i=1}^{r} C(d_i). \tag{5}$$

Note that the existence of a polyhedron whose faces are of degree $d_1, \dots, d_r$ is not assumed.

Proof. Fix $r \geq 9$ and p. Of the sequences $(d_1, \dots, d_r)$ satisfying the conditions (3) and (4), choose one that attains the maximal capacity and denote it by $(d'_1, \dots, d'_r)$. Clearly, it suffices to show the inequality (5) for these "maximal sequences."

Claim. *There is* k *such that*

$$r-1 = d'_1 = \cdots = d'_{k-1} \geq d'_k > d'_{k+1} = \cdots = d'_r = 3.$$

We prove the claim by contradiction; assume that $r-1 > d'_i \geq d'_j > 3$ for some i and j. Consider replacing (d'_i, d'_j) by (d'_i+1, d'_j-1). The resulting sequence would still satisfy the conditions (3) and (4), and the maximality of $\sum_{i=1}^{r} C(d'_i)$ implies

$$C(d'_i) + C(d'_j) \geq C(d'_i+1) + C(d'_j-1).$$

As we can see from the table below, this can occur only if

$$(d'_i, d'_j) = (5,5) \text{ or } (6,5).$$

d	3		4		5		6		7		8		9
$C(d)$	0		8		20		30		42		56		72
$C(d+1)-C(d)$		8		12		10		12		14		16	

However, since

$$C(5) + C(5) = 40 < 42 = C(7) + C(3),$$
$$C(6) + C(5) = 50 < 56 = C(8) + C(3),$$

replacing the pair $(5,5)$ or $(6,5)$ respectively by $(7,3)$ or $(8,3)$ would produce a sequence of greater capacity than $(d'_1, \dots, d'_r)$; the replacement is allowed for $r \geq 9$. This contradiction proves the claim.

Now we go back to the proof of the lemma. We write

$$\begin{aligned} d'_1 = \cdots = d'_{k-1} &= s+3, \\ d'_k &= t+3, \end{aligned}$$

thus

$$0 \le t \le s = r-4.$$

By the above claim, we can compute p and the capacity of $(d'_1, \ldots, d'_r)$ as follows:

$$\begin{aligned} p &= 2 + \frac{1}{2}\sum_{i=1}^{r} d'_i - r \\ &= 2 + \frac{1}{2}\{(k-1)(r-1) + (t+3) + 3(r-k)\} - r \\ &= \frac{1}{2}(ks+t+8), \end{aligned}$$

$$\sum_{i=1}^{r} C(d'_i) = (k-1)(s+3)(s+2) + (t+3)(t+2).$$

Hence we have

$$\begin{aligned} &4\left(p(p-1) - \sum_{i=1}^{r} C(d'_i)\right) \\ &= (ks+t+8)(ks+t+6) - 4(k-1)(s+3)(s+2) - 4(t+3)(t+2) \\ &= \left(ks+t-2s-3-\frac{12}{s}\right)^2 + 4t(s-t) + 24\frac{t}{s} + 8s - 9 - \frac{72}{s} - \frac{144}{s^2}. \end{aligned}$$

It is easy to verify that

$$8s - 9 - \frac{72}{s} - \frac{144}{s^2} > 0$$

when $s = r-4 \ge 5$. Therefore we have

$$p(p-1) > \sum_{i=1}^{r} C(d'_i).$$

The lemma has been proved. □

Next consider the case where $r < 9$. Running a computer program shows that there are twelve sequences $(d_1, \ldots, d_r)$ which satisfy (3) and

$$p(p-1) \le \sum_{i=1}^{r} C(d_i).$$

Of the twelve sequences, only $(5,5,4,4,3,3)$ and $(6,5,5,5,3,3,3)$ satisfy the following two necessary conditions for polyhedra:

$$p \ge d_1 + d_2 + d_3 - 6$$

since each pair of faces share at most two vertices, and

$$3p \le \sum_{i=1}^{r} d_i$$

since every vertex is incident with at least three faces. For the two sequences given above, which correspond to the polyhedra shown in Figure 1, the equality in (2) holds. This completes the proof of Theorem 1. □

The following corollary of Theorem 1 will be needed in the proof of the Main Theorem.

Corollary 3. *Let P be a polyhedron. Then there exists a vertex $v \in V(P)$ such that*

$$\sum_{v \in f \in F(P)} D(f) \leq |V(P)| - 2. \tag{6}$$

Proof. One can easily check the statement for the two polyhedra shown in Figure 1; we assume that P is neither of the two polyhedra. If

$$|V(P)| - 1 \leq \sum_{v \in f} D(f)$$

for all $v \in V(P)$, then we would have

$$\begin{aligned} |V(P)|(|V(P)| - 1) &\leq \sum_{v \in V(P)} \sum_{v \in f} D(f) \\ &= \sum_{f \in F(P)} C(f). \end{aligned}$$

However this contradicts Theorem 1. Therefore there must be a vertex $v \in V(P)$ satisfying (6). □

3. Proof of the Main Theorem

Let M be a 3-manifold obtained from convex ideal polyhedra $P_1, \ldots, P_n$ by identifying the faces in pairs. A pair of faces which are to be identified are called partner faces of each other. We denote by $F^{(i)}(P_j)$ the set of faces $f \in F(P_j)$ whose partner face belongs to $F(P_i)$.

We first introduce what we call a "cone decomposition" of a convex polyhedron. Let P be a convex polyhedron, and $v \in V(P)$ a vertex of P. Divide those faces of P not containing v into triangles (in any way you like), and the polyhedron P is decomposed into the cones of these triangles from the vertex v. This decomposition is called a cone decomposition of P from v. Note that taking arbitrary cone decompositions of $P_1, \ldots, P_n$ does not in general result in a proper decomposition of M, for the induced triangulations of partner faces may not coincide when the faces are identified; although this difficulty can be easily avoided if we are allowed to insert degenerate (flattened) ideal tetrahedra between such incompatible pairs of partner faces.

We will show, under the assumption that each face of P_i ($i = 1, \ldots, n-1$) is identified with a face of P_n, that the above incompatibility does not occur if we carefully choose cone decompositions of $P_1, \ldots, P_n$. For $i = 1, \ldots, n-1$, we choose a vertex $v_i \in V(P_i)$ given by Corollary 3 so that

$$\sum_{v_i \in f \in F(P_i)} D(f) \leq |V(P_i)| - 2,$$

and take a cone decomposition of P_i from v_i.

Now consider choosing a vertex $v_n \in V(P_n)$, and taking a cone decomposition of P_n from v_n. Recall that, when taking a cone decomposition of a polyhedron P from a vertex v, we can freely triangulate those faces of P not containing v. Hence, by choosing a cone decomposition of P_i and one of P_n appropriately, we can make the induced triangulations of the faces of P_i and of P_n compatible; unless there are partner faces $f \in F(P_i)$ and $f' \in F(P_n)$ such that $v_i \in f$, $v_n \in f'$, and either

(1) $\deg f = \deg f' = 4$, and v_i is adjacent to v_n when f and f' are identified, or
(2) $\deg f = \deg f' \geq 5$, and v_i does not correspond to v_n under the identification.

Let A_i denote the set of vertices $v_n \in V(P_n)$ from which every cone decomposition of P_n causes an incompatibility problem of facial triangulations between P_i and P_n. Then by the above we have

$$|A_i| \leq \sum_{v_i \in f} D(f) \leq |V(P_i)| - 2.$$

We also need to choose a cone decomposition of P_n from v_n so that the induced triangulations of each pair of partner faces of P_n coincide under the identification. This can be done unless there exist a pair of partner faces $f, f' \in F^{(n)}(P_n)$ sharing the vertex v_n such that either

(1) $\deg f = \deg f' = 4$, and the gluing map sends v_n to one of the vertices of f' adjacent to v_n, or
(2) $\deg f = \deg f' \geq 5$, and the identification does not map v_n to itself.

Let A_n denote the set of vertices $v_n \in V(P_n)$ from which every cone decomposition of P_n induces incompatible triangulations on some pair of partner faces of P_n. Since two faces can share at most two vertices, we have

$$|A_n| \leq \sum_{d \geq 4} |F_d^{(n)}(P_n)|,$$

where $F_d^{(n)}(P_n)$ denotes the set of faces $f \in F^{(n)}(P_n)$ of degree d.

To prove the theorem, it suffices to show that there exists a vertex of P_n not contained in any of the sets $A_1, \ldots, A_n$. We show this by proving $|V(P_n)| > \sum_{i=1}^{n} |A_i|$ as follows:

$$\begin{aligned} |V(P_n)| - 2 &= |E(P_n)| - |F(P_n)| \\ &= \sum_{f \in F(P_n)} \left(\frac{1}{2} \deg f - 1 \right) \\ &= \sum_{i=1}^{n} \sum_{f \in F^{(i)}(P_n)} \left(\frac{1}{2} \deg f - 1 \right). \end{aligned}$$

Here, for $i = 1, \ldots, n-1$, we have

$$\begin{aligned} \sum_{f \in F^{(i)}(P_n)} \left(\frac{1}{2} \deg f - 1 \right) &= \sum_{f \in F(P_i)} \left(\frac{1}{2} \deg f - 1 \right) \\ &= |V(P_i)| - 2 \\ &\geq |A_i|, \end{aligned}$$

and

$$\sum_{f \in F^{(n)}(P_n)} \left(\frac{1}{2}\deg f - 1\right) = \sum_{d \geq 3} \left(\frac{d}{2} - 1\right) |F_d^{(n)}(P_n)|$$
$$\geq \sum_{d \geq 4} |F_d^{(n)}(P_n)|$$
$$\geq |A_n|.$$

Therefore we have

$$|V(P_n)| - 2 \geq \sum_{i=1}^{n} |A_i|.$$

This completes the proof of the Main Theorem.

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