

ON RATIONAL INVARIANTS OF THE GROUP E_6

A. V. ILTYAKOV

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ABSTRACT. We prove rationality of the field of invariants in several variables of a minimal irreducible representation of a simple algebraic group of type E_6 over an algebraically closed field of characteristic zero.

This note completes the main result of [3], where the reader can find an extensive introduction and basic notations.

Let $\mathbf{A} = H(\mathbf{O}_3)$ be an Albert algebra [4, p.21] over an algebraically closed field F of characteristic 0. Any element $x \in \mathbf{A}$ satisfies a cubic equation over F

$$(1) \quad x^3 - x^2 t_1(x) + x t_2(x) - e t_3(x) = 0,$$

where e is the unit of \mathbf{A} , $t_1(x)$ is the trace $tr(x)$ of x and $t_3(x)$ is the cubic norm $N(x)$ of x .

The group of norm-preserving linear transformations

$$M = \{g \in \text{End}(\mathbf{A}) \mid g \text{ is invertible, } \forall x \in \mathbf{A} \ N(x \cdot g) = N(x)\}$$

is a simple algebraic group of type E_6 [1] and \mathbf{A} is an irreducible M -module of dimension 27 [5, th.14]. The form $tr(xy)$ is nondegenerate. For $g \in \text{End}(\mathbf{A})$ denote by g^* the transposed transformation with respect to this form. We may consider \mathbf{A} as an M -module \mathbf{A}^* under the action $x \rightarrow x \cdot g^T$, where $g \rightarrow g^T = (g^*)^{-1}$ is an (outer) automorphism of M [4, p.370, ex.2] (in other words, \mathbf{A}^* is the module contragredient to \mathbf{A}). Let $\mathbf{V}_{k,m} = k\mathbf{A} \oplus m\mathbf{A}^*$, where $k\mathbf{A}$ ($m\mathbf{A}^*$) is the direct sum of k (m) copies of \mathbf{A} (\mathbf{A}^*) and M acts diagonally.

Theorem 1. *For any $k, m \geq 0$ the field of rational invariants $F(\mathbf{V}_{k,m})^M$ is purely transcendental over F .*

Proof. Since $F(\mathbf{V}_{k,m})^M \simeq F(\mathbf{V}_{m,k})^M$, we can assume that $k \geq 1$. Using the slice method [2] (see also, for example, [8]), we shall show that this field is isomorphic to a simple transcendental extension of $F(n\mathbf{A})^G$, where $n = m + k - 1$ and G is the group of automorphisms $\text{Aut}(\mathbf{A})$. Since rationality of the latter field is obtained in [3], this will prove the statement. In particular, the transcendence degree of this

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During 1993/94 a Becario de FICYT at the University of Oviedo, Asturias, Spain.

field is computed in the following way (cf. [3]):

$$\text{tr. deg}_F F(\mathbf{V}_{k,m})^M = \begin{cases} 30 + 27(m + k - 4), & m + k \geq 4, \\ 11, & m + k = 3, \\ 3, & m + k = 2, \\ 1, & m + k = 1. \end{cases}$$

Lemma 1 ([5, th.7], [4, p.244]). 1. *If $a \in \mathbf{A}$ and $N(a) = 1$, then there is $\eta \in M$ such that $a \cdot \eta = e$.*

2. *If $\phi \in M$ and $e \cdot \phi = e$, then ϕ is an automorphism of \mathbf{A} .*

From the lemma it follows that the subspace

$$E = \{\bar{a} = (a_1, \dots, a_k, b_1, \dots, b_m) \in \mathbf{V}_{k,m} \mid a_k \in eF\}$$

is a slice of the M -module $\mathbf{V}_{k,m}$ [2, sect.3]. Hence, $F(\mathbf{V}_{k,m})^M \simeq F(E)^{N(E)}$, where $N(E) = \{\phi \in M \mid E \cdot \phi \subseteq E\}$. Also, $N(E) = G \times C_3$, where C_3 is the group of linear transformations $R_x : y \rightarrow yx$, where $x = e\xi$, $\xi \in F$ and $\xi^3 = 1$.

Notice that $E = eF \oplus \mathbf{V}_{k-1,m}$. Denote by t the coordinate function of the first summand and by y_i (z_i) the projector from $k\mathbf{A}$ ($m\mathbf{A}^*$) onto the i -th summand (generic elements of \mathbf{A} [3]). Consider a monomorphism $\psi : F(\mathbf{V}_{k-1,m}) \rightarrow F(E)$: if $f = f(y_2, \dots, y_k, z_1, \dots, z_m)$, then $\psi(f) = f(y_2/t, \dots, y_k/t, z_1t, \dots, z_mt)$. Since $R_x^\tau = R_{x^{-1}}$ for $R_x \in C_3$, the image K of $F(\mathbf{V}_{k-1,m})^G$ lies in $F(E)^{N(E)} = (F(E)^G)^{C_3}$. Observe that $F(E)$ can be considered as the field of rational functions of the G -module $\mathbf{V}_{k-1,m} \otimes_F F(t)$ over $F(t)$, where the generic elements are $y_2/t, \dots, y_k/t, z_1t, \dots, z_mt$; hence, $F(E)^G = K(t)$. Since for the extension degree we have $|F(E)^{N(E)} : F(E)^G| = 3$, we get $K(t^3) = F(E)^{N(E)} \simeq F(\mathbf{V}_{k,m})^M$. \square

It is known that the field of invariants $F(k\mathbf{A})^G$ is generated by trace polynomials [3]. Working out this proof, we shall describe generators of the field $F(\mathbf{V}_{k,m})^M$ in a similar way. Let $N(x, y, z)$ be the linearization of $N(x)$ ($N(x, x, x) = 3N(x)$) and let $x \rightarrow x^\#$ be a quadratic operation such that $\text{tr}(x^\#y) = N(x, x, y)$ [4, 6]. Also, $x \times y = (x + y)^\# - x^\# - y^\#$. Consider a pair of subspaces $\Delta, \Delta^* \subseteq \mathbf{A} \otimes_F F[\mathbf{V}_{k,m}]$ which are minimal with the following properties:

1. $y_1, \dots, y_k \in \Delta, z_1, \dots, z_m \in \Delta^*$;
2. if $u, v \in \Delta$ ($u, v \in \Delta^*$), then $u \times v \in \Delta^*$ (Δ);

Theorem 2. *The field $F(\mathbf{V}_{k,m})^M$ is generated by elements $N(u)$, $u \in \Delta$.*

By the definition of $x \rightarrow x^\#$, for any $g \in M$ we have $(x \cdot g)^\# = x^\# \cdot g^\tau$. The next statement can be easily checked by induction.

Lemma 2. *Let $u \in \Delta$ (Δ^*). Then for any $g \in M$ the polynomial*

$$u(y_1 \cdot g, \dots, y_k \cdot g, z_1 \cdot g^\tau, \dots, z_m \cdot g^\tau)$$

is equal to $u \cdot g$ (respectively, $u \cdot g^\tau$).

In particular, this yields $N(u) \in F(\mathbf{V}_{k,m})^M$ for any $u \in \Delta$. Since M is a simple linear algebraic group, the field $F(\mathbf{V}_{k,m})^M$ is generated by homogeneous polynomial invariants [7, sect.3.3]. Take such an invariant $f = f(y_1, \dots, y_k, z_1, \dots, z_m)$ and put $p = \sum_{i=1}^k \text{deg}_{y_i}(f)$, $q = \sum_{i=1}^m \text{deg}_{z_i}(f)$. Recall that E is a slice and, therefore, the mapping $\pi : f \rightarrow f|_E$ is a monomorphism from the algebra of polynomial invariants $F[\mathbf{V}_{k,m}]^M$ to $F(E)^{N(E)}$ and its image generates this field. Denote $\Omega =$

$alg_F\{N(u) \mid u \in \Delta\} \subseteq F[\mathbf{V}_{k,m}]^M$. To prove the theorem it suffices to show that $\pi(\tilde{f})$ belongs to the field K generated by $\pi(\Omega)$.

Observe that $t^3 = \pi(N(y_1)) \in K$. Also, $\pi(f) = ht^{p-q}$, where

$$h = f(e, y_2/t, \dots, y_k/t, z_1t, \dots, z_mt) = \psi(\tilde{f})$$

and $\tilde{f} = f|_{y_1=e} \in F(\mathbf{V}_{k-1,m})^G$. Recall that $\psi(\tilde{f}) \in F(E)^{N(E)}$; hence, t^{p-q} also lies in this field. Therefore, $r = (p - q)/3$ is an integer and $t^{p-q} \in K$. In particular, $\psi(\tilde{g})$ is defined and belongs to K for any $g \in \Omega$ and, therefore, for any $g \in F(\Omega)$. If we show that $\tilde{f} = \tilde{g}$ for some $g \in F(\Omega)$, then we shall get $h = \psi(\tilde{g}) \in K$ and prove the theorem.

The field $F(\mathbf{V}_{k-1,m})^G$ is generated by polynomials $tr(u)$, where $u \in B_{k+m-1} = alg_F\{y_2, \dots, y_k, z_1, \dots, z_m\}$ [3]. It suffices to show that $tr(u) = \tilde{g}$ for an appropriate $g \in F(\Omega)$. Let $\Delta\Omega$ be the Ω -module generated by Δ .

Lemma 3. *For any $u \in B_{k+m-1}$ there is $h_u \in \Delta\Omega$ such that $h_u|_{y_1=e} = u$.*

Proof. We shall need the following identities, which follow from [6, eq.(11),(21)]:

$$(2) \quad a = -a \times e + tr(a) \cdot e ,$$

$$(3) \quad a^2 = (a \times e)^\# - N(a, a, e)e .$$

Denote $\omega(a, b) = -a \times b^\# + N(a, b^\#, b^\#)b$. By (2), if $a \in \Delta^*$ and $b \in \Delta$, then $\omega(a, b) \in \Delta\Omega$ and $\omega(a, e) = a$. We assume that u is a monomial and carry out an induction on its degree. First, put $h_{y_i} = y_i$, $2 \leq i \leq k$, and $h_{z_l} = \omega(z_l, y_1)$, $1 \leq l \leq m$. Now, suppose that $deg(u) = 2$, i.e. u has the form $u_1 = y_i y_j$, $u_2 = z_s y_j$ or $u_3 = z_s z_t$. Using the linearized form of (3), we may put

$$h_{u_1} = 1/2(y_i \times y_1) \times (y_j \times y_1) - N(y_i, y_j, y_1)y_1 .$$

Next,

$$h_{u_2} = h_{u_1}|_{y_i=\omega(z_s, y_1)}, \quad h_{u_3} = h_{u_2}|_{y_j=\omega(z_t, y_1)} .$$

We can assume that the lemma holds for an arbitrary monomial $u \in B_{k+m-1}$ of degree $< n$ and for arbitrary k, m . Every monomial u , $deg(u) = n \geq 3$, can be represented in the form $u = v|_{y_{k+1}=w}$, where $deg(v), deg(w) < n$. Hence,

$$h_u = h_v|_{y_{k+1}=h_w} .$$

To conclude the proof of Theorem 2 we may put $g = N(h_u, y_1, y_1)$. □

Recall that Ω lies in the algebra of polynomial invariants $F[\mathbf{V}_{k,m}]^M$, and it looks probable that this algebra is generated by elements $N(u)$, $N(v)$, $tr(uv)$, where $u \in \Delta$, $v \in \Delta^*$.

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INSTITUTE OF MATHEMATICS, SIBERIAN BRANCH OF RUSSIAN ACADEMY OF SCIENCES, NOVOSIBIRSK, 630090, RUSSIA

Current address: School of Mathematics and Statistics, University of Sydney, 2006 New South Wales, Australia