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ON COMPLETELY HYPEREXPANSIVE OPERATORS

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ABSTRACT. We introduce and discuss a class of operators, to be referred to as the class of completely hyperexpansive operators, which is in some sense antithetical to the class of contractive subnormals. The new class is intimately related to the theory of negative definite functions on abelian semigroups. The known interplay between positive and negative definite functions from the theory of harmonic analysis on semigroups can be exploited to reveal some interesting connections between subnormals and completely hyperexpansive operators.

If $\mathbb{H}$ is a complex infinite-dimensional separable Hilbert space, then $\mathbb{B}(\mathbb{H})$ will denote the set of bounded linear operators on $\mathbb{H}$. Recall that S in $\mathbb{B}(\mathbb{H})$ is said to be *subnormal* if there exist a Hilbert space $\mathbb{K}$ containing $\mathbb{H}$ and a normal operator N in $\mathbb{B}(\mathbb{K})$ such that $N\mathbb{H} \subset \mathbb{H}$ and $N/\mathbb{H} = S$. For all of the elementary results pertaining to subnormals (and related classes of operators such as hyponormals) that are stated below without proof, the reader is referred to [Co]. If $\{e_n\}_{n\geq 0}$ is an orthonormal basis for $\mathbb{H}$, then a weighted shift operator T on $\mathbb{H}$ with the weight sequence $\{\alpha_n\}_{n\geq 0}$ is defined through the relations $Te_n = \alpha_n e_{n+1}$ $(n \geq 0)$. We will always assume that $\alpha_n > 0$ for all n. An excellent reference for the basic properties of weighted shifts is [S]. We will often use the notation $T : \{\alpha_n\}$ to indicate a weighted shift. Of particular interest to us are the shifts $U : \{1\}$ (the unilateral shift), $B : \{\sqrt{n+1}/\sqrt{n+2}\}$ (the Bergman shift) and $D : \{\sqrt{n+2}/\sqrt{n+1}\}$ (the Dirichlet shift). The shifts U and B are subnormal with U in fact being an isometry; while D is a 2-isometry ($I - 2D^*D + D^{*2}D^2 = 0$, I and 0 being the identity and zero operator respectively).

It is the purpose of this note to introduce and discuss a class of operators that is in some sense antithetical to the class of contractive subnormals. The connections between the theory of positive definite functions on abelian semigroups and the theory of subnormals are well-known and will be touched upon briefly in the sequel. The new class of operators, to be referred to as the class of completely hyperexpansive operators, is intimately related to the theory of negative definite functions on abelian semigroups. The shift U and the shift D are rather special examples of the new class. The known connections between positive and negative definite functions will allow us to correlate U and D as well as B and D in meaningful ways. Using the idea of the Laplace Transform, we will be able to associate with every hyperex-

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pansive weighted shift a one-parameter family of contractive subnormal weighted shifts. The Lévy-Khinchin representation theory from harmonic analysis on semigroups will play for the new class a role analogous to that of the Hausdorff Moment Problem for subnormals. The arguments to follow rely heavily on the theory of positive and negative definite functions as expounded in [B-C-R2].

Recall that a real-valued function φ on the semigroup $\mathbb{N}$ of non-negative integers is said to be *positive definite* if $\sum_{i,j=1}^{n} c_i c_j \varphi(s_i+s_j) \geq 0$ for all $n \geq 1$, $\{s_1, \ldots, s_n\} \subset \mathbb{N}$ and $\{c_1, \ldots, c_n\} \subset \mathbb{R}$. A real map ψ on $\mathbb{N}$ is said to be *negative definite* if $\sum_{i,j=1}^{n} c_i c_j \psi(s_i + s_j) \leq 0$ for all $n \geq 2$, $\{s_1, \ldots, s_n\} \subset \mathbb{N}$ and for $\{c_1, \ldots, c_n\} \subset \mathbb{R}$ such that $\sum_{i=1}^{n} c_i = 0$.

We let the difference operator ∇ act on φ through the formula $(\nabla\varphi)(s) = \varphi(s) - \varphi(s+1)$. The relations $\nabla^0\varphi = \varphi$ and $\nabla^n\varphi = \nabla\nabla^{n-1}\varphi$ inductively define ∇^n for all $n \geq 0$. One can similarly define the powers of the corresponding difference operator Δ given by $(\Delta\varphi)(s) = \varphi(s+1) - \varphi(s)$. A non-negative map φ on $\mathbb{N}$ is said to be *completely monotone* if $(\nabla^n\varphi)(s) \geq 0$ for all $s, n \geq 0$. A real map ψ on $\mathbb{N}$ is said to be *completely alternating* if $(\nabla^n\varphi)(s) \leq 0$ for all $s \geq 0, n \geq 1$. A Radon measure μ on a subset X of $\mathbb{R}$ will be understood to be a Borel measure μ satisfying (i) $\mu(C) < \infty$ for each compact subset C of X, (ii) $\mu(B) = \sup\{\mu(C) : C \subset B, C$ compact$\}$ for each Borel set B in X. A finite Radon measure is easily seen to be a regular Borel measure. Propositions 1 and 2 below are specialized (one-dimensional) versions of Proposition 6.11 and Proposition 6.12 in Chapter 4 of [B-C-R2].

Proposition 1. *For $\varphi : \mathbb{N} \to \mathbb{R}$, the following are equivalent.*

(i) *φ is completely monotone.*

(ii) *$\sum_{0\leq p\leq n} (-1)^p \binom{n}{p} \varphi(m+p) \geq 0$ for all $m, n \geq 0$.*

(iii) *There exists a positive Radon measure μ on $[0,1]$ such that*

$$\varphi(n) = \int_{[0,1]} x^n \, d\mu(x) \quad \text{for all } n \geq 0. \tag{A}$$

Proposition 2. *For $\psi : \mathbb{N} \to \mathbb{R}$, the following are equivalent.*

(i) *ψ is completely alternating.*

(ii) *$\sum_{0\leq p\leq n} (-1)^p \binom{n}{p} \psi(m+p) \leq 0$ for all $m \geq 0, n \geq 1$.*

(iii) *There exist a real a, a non-negative real b and a positive Radon measure ν on $[0,1] - \{1\} = [0,1)$ such that*

$$\psi(n) = a + nb + \int_{[0,1)} (1 - x^n) \, d\nu(x) \quad \text{for all } n \geq 1. \tag{B}$$

(The representation holds for $n = 0$ as well, provided one interprets x^0 to be 1 for all x in $[0,1)$ and employs the usual measure-theoretic convention $0.\infty = 0$.)

Remark 1. The result in Proposition 1 is the solution of the classical Hausdorff Moment Problem. The multidimensional version of Proposition 2 as stated in [B-C-R2] was first obtained as a special case of the Lévy-Khinchin representation theory on abelian semigroups in [B-C-R1]. For our purposes, it will be instructive to derive representation (B) from (A). Note that $\psi : \mathbb{N} \to \mathbb{R}$ is completely alternating if and only if $\Delta\psi$ is completely monotone (cf. [B-C-R2], Ch. 4, Lemma 6.3). Letting

$\varphi(n) = (\Delta\psi)(n) = \psi(n+1) - \psi(n)$, we have, for all $n \geq 1$,

$$\begin{aligned}\psi(n) &= \psi(0) + \sum_{0\leq k\leq n-1} (\psi(k+1) - \psi(k)) = \psi(0) + \sum_{0\leq k\leq n} \varphi(k)\\ &= \psi(0) + \sum_{0\leq k\leq n-1} \int_{[0,1]} x^k\, d\mu(x) = \psi(0) + \int_{[0,1]} \left(\sum_{0\leq k\leq n-1} x^k \right) d\mu(x)\\ &= \psi(0) + n\mu(\{1\}) + \int_{[0,1)} (1-x^n) \frac{d\mu(x)}{1-x}.\end{aligned}$$

Thus a, b and $d\nu(x)$ in (B) can be identified with $\psi(0), \mu(\{1\})$ and $d\mu(x)/(1-x)$, respectively. We emphasize that $\nu([0,1))$ may not be finite, while μ in Proposition 1 is a finite positive regular Borel measure on $[0,1]$.

The operator-theoretic significance of Proposition 1 was reflected in Jim Agler's criterion for subnormality [A]: T in $\mathbb{B}(\mathbb{H})$ is a subnormal contraction if and only if

$$\text{(C)} \qquad \sum_{0\leq p\leq n} (-1)^p \binom{n}{p} T^{*p}T^p \geq 0 \quad \text{for all } n \geq 0.$$

The equivalence of (C) with the complete monotonicity of $n \to \|T^n h\|^2$, for any h in $\mathbb{H}$, was emphasized in [At1]. At this stage we point out that the set of completely monotone maps is known to be an extreme subset of the set of bounded positive definite maps on $\mathbb{N}$ ([B-C-R2], Ch. 4, Thm. 6.5). We now begin to explore the operator-theoretic significance of Proposition 2.

Definition. An operator T in $\mathbb{B}(\mathbb{H})$ is *completely hyperexpansive* if

$$\text{(D)} \qquad \sum_{0\leq p\leq n} (-1)^p \binom{n}{p} T^{*p}T^p \leq 0 \quad \text{for all } n \geq 1.$$

Remark 2. The first among the countable conditions in (D) is $I - T^*T \leq 0$, guaranteeing that $\|T\| \geq 1$ so that T is an 'expansion'. In analogy with [A], we may refer to any T satisfying the first n conditions in (D) as an n-hyperexpansion. The unilateral shift U, being an isometry, is clearly completely hyperexpansive. It was noted in [R] that any operator T satisfying $I - 2T^*T - T^{*2}T^2 \leq 0$ is actually a 2-hyperexpansion, that is, T satisfies $I - T^*T \leq 0$ as well. In particular, the Dirichlet shift D, being a 2-isometry, is completely hyperexpansive. If the left side of the inequality in (D) is denoted by $B_n(T)$, then one has, for any m in $\mathbb{N}$ and h in $\mathbb{H}$, $\langle T^{*m}B_n(T)T^m h, h\rangle \leq 0$, that is,

$$\sum_{0\leq p\leq n} (-1)^p \binom{n}{p} \psi(m+p) \leq 0 \quad \text{for } m \geq 0, n \geq 1, \text{where } \psi(n) = \|T^n h\|^2.$$

In view of Proposition 2 and Remark 1, ψ is completely alternating and

$$\text{(E)} \qquad \|T^n h\|^2 = \|h\|^2 + n\mu_h(\{1\}) + \int_{[0,1)} (1-x^n) \frac{d\mu_h(x)}{1-x},$$

where μ_h is a positive regular Borel measure on $[0,1]$. Clearly, T is hyperexpansive if and only if (E) holds for any h in $\mathbb{H}$ and $n \geq 1$. We may refer to (E) as the Lévy-Khinchin representation for the pair (T, h) and to $d\nu_h(x) = d\mu_h(x)/(1-x)$ as the Lévy measure for (T, h). (Consult [B-C-R2] for the justification of this terminology.) An immediate consequence of (E) is that $\|T^n h\|^2 \geq \|h\|^2$ for all h

in $\mathbb{H}$ and $n \geq 0$. At this stage we point out that the set of completely alternating maps is known to be an extreme subset of the set of all negative definite functions on $\mathbb{N}$ that are bounded below (see [B-C-R2], Ch. 4, Thm. 6.7), and the property $\psi(n) \geq \psi(0)$ actually holds for any negative definite function ψ on $\mathbb{N}$ bounded below (see [B-C-R2], Ch. 4. 3.2). Using (E), one can verify that any positive integral power of a completely hyperexpansive operator is completely hyperexpansive. Indeed, if T is completely hyperexpansive and $k \geq 1$, then

$$\begin{aligned}\|(T^k)^n h\|^2 = \|T^{nk}h\|^2 &= \|h\|^2 + nk\mu_h(\{1\}) + \int_{[0,1)} (1-x^{nk})\frac{d\mu_h(x)}{1-x} \\ &= \|h\|^2 + n(k\mu(\{1\})) + \int_{[0,1)} (1-y^n)\frac{d\mu'_h(y)}{1-y^{1/k}}\end{aligned}$$

for appropriate μ_h and μ'_h, so that $n \to \|(T^k)^n h\|^2$ is completely alternating.

We associate with every weighted shift $T : \{\alpha_n\}$ a sequence $\{\beta_n(T)\}$ defined by $\beta_0(T) = 1, \beta_1(T) = \alpha_0^2$, and $\beta_n(T) = \alpha_0^2\alpha_1^2 \cdots \alpha_{n-1}^2$ $(n \geq 2)$. Occasionally, we may simply write β_n for $\beta_n(T)$. Recall that T is a contractive subnormal weighted shift if and only if there exists a Borel probability measure μ on $[0,1]$ such that $(\|T^n e_0\|^2 =)\beta_n = \int_{[0,1]} x^n \, d\mu(x)$ for all $n \geq 0$. If T is a completely hyperexpansive weighted shift, then one clearly has

$$\text{(F)} \qquad \begin{aligned}(\|T^n e_0\|^2 =)\beta_n &= 1 + n\mu(\{1\}) + \int_{[0,1)} (1-x^n)\frac{d\mu(x)}{1-x} \\ &= 1 + \int_{[0,1]} (1 + \cdots + x^{n-1}) \, d\mu(x) \quad \text{for all } n \geq 1,\end{aligned}$$

where μ is a positive regular Borel measure on $[0,1]$.

Proposition 3. *If $T : \{\alpha_n\}$ is a weighted shift such that $n \to \beta_n$ is completely alternating on $\mathbb{N}$ (so that (F) holds), then T is completely hyperexpansive.*

Proof. Suppose $n \to \beta_n$ is completely alternating. Since T is a weighted shift, it suffices to check that

$$\sum_{0 \leq p \leq n} (-1)^p \binom{n}{p} \|T^p e_m\|^2 \leq 0 \quad \text{for any } m \geq 0,\ n \geq 1.$$

Since $e_m = a_m T^m e_0$ for some positive a_m, we need only verify

$$\sum_{0 \leq p \leq n} (-1)^p \binom{n}{p} \|T^{p+m} e_0\|^2 \leq 0 \quad \text{for } m \geq 0,\ n \geq 1;$$

that is,

$$\sum_{0 \leq p \leq n} (-1)^p \binom{n}{p} \beta_{m+p} \leq 0 \quad \text{for } m \geq 0,\ n \geq 1.$$

But this follows from the equivalence of (i) and (ii) in Proposition 2. □

Remark 3. The part $a+nb$ $(= 1+n\mu(\{1\}))$ in the representation of β_n can actually be looked upon as $\beta_n(T')$, where $T' : \{(1+(n+1)\mu(\{1\}))^{1/2}/(1+n\mu(\{1\}))^{1/2}\}$ is clearly a 2-isometry. Thus the integral part $\int_{[0,1)}(1-x^n)\,d\mu(x)/(1-x)$ in the representation for β_n can be regarded as a "non-2-isometric" perturbation of the "2-isometric" part $1 + n\mu(\{1\})$. Proposition 3 offers us an easy way of generating completely hyperexpansive weighted shifts by making various choices for

$d\nu(x)$ $(= d\mu(x)/(1-x))$. From our earlier observations, T is completely hyperexpansive if and only if $n \to (\|T^{n+1}h\|^2 - \|T^n h\|^2)$ is completely monotone for any h in $\mathbb{H}$. In the context of weighted shifts, we are thus led to consider $\{\|T^{n+1}e_0\|^2 - \|T^n e_0\|^2\} = \{\beta_{n+1} - \beta_n\}$. If we let $\beta_{n+1}(T) - \beta_n(T) = \beta_n(U)$, where U is the unilateral shift, then $\beta_n(T) = n+1$. But this is precisely $\beta_n(D)$, where D is the Dirichlet shift! In [A], Agler constructed a sequence of functional Hilbert spaces $\mathbb{H}_n$ such that the model-theoretic properties of the corresponding (subnormal) multiplication operators $M_z^{(n)}$ tied up nicely with his criterion for subnormality mentioned earlier. In fact, $M_z^{(1)} = U$ and $M_z^{(2)} = B$. It was noted in [At2] that, for $k \geq 2$, the measure μ $(= \mu_k)$ in Proposition 1 corresponding to $\varphi(n) = \beta_n(M_z^{(k)})$ is given by $d\mu_k(x) = (1/(k-2)!)(1-x)^{k-2}\,dx$, where dx is the 1-dimensional Lebesgue measure on $[0,1]$. If $N^{(k)}$ denotes the completely hyperexpansive weighted shift associated with $M_z^{(k)}$ through Δ, that is, $\Delta\beta_n(N^{(k)}) = \beta_n(M_z^{(k)})$, then $\beta_n(N^{(k)}) = 1 + n\mu_k(\{1\}) + \int_{[0,1)}(1-x^n)\,d\mu_k(x)/(1-x)$. For $k \geq 2$, $(b =)$ $\mu_k(\{1\}) = 0$ so that

$$\beta_n(N^{(k)}) = 1 + (1/(k-2)!)\int_{[0,1)}(1-x^n)(1-x)^{k-3}\,dx.$$

It is interesting to note that the integral

$$\int_{[0,1)} d\nu_k(x) = \int_{[0,1)} (1-x)^{k-3}/(k-2)!\,dx$$

is infinite for $k = 2$, and finite for any $k \geq 3$. In view of Theorem 3.20 in Chapter 4 of [B-C-R2], this implies that the sequence $\{\beta_n(N^{(k)})\}$ is unbounded for $k = 2$, and bounded for $k \geq 3$. (Note that $(\beta_n(N^{(2)}) - 1)$ is simply the nth partial sum of the harmonic series.)

It is well-known that the weight sequence corresponding to a subnormal (and indeed a hyponormal) weighted shift is non-decreasing. The following proposition shows in particular that the opposite is true for a completely hyperexpansive weighted shift.

Proposition 4. *Let $T : \{\alpha_n\}$ be a completely hyperexpansive weighted shift. Then $\alpha_n \geq 1$ for all $n \geq 0$ so that $\beta_n \geq 1$ for all $n \geq 0$. The sequence $\{\alpha_n\}$ is non-increasing (so that $\|T\| = \alpha_0$), and moreover satisfies*

$$\alpha_{n+1}^2(\alpha_{n+2}^2 - 1)(\alpha_n^2 - 1) \geq \alpha_n^2(\alpha_{n+1}^2 - 1)^2 \quad \textit{for all } n \geq 0.$$

Proof. Let $T : \{\alpha_n\}$ be completely hyperexpansive. Then $\alpha_n = \|Te_n\| \geq \|e_n\| = 1$ for all $n \geq 0$, so that $\beta_n \geq 1$ for all $n \geq 0$. To check that $\{\alpha_n\}$ is non-increasing, we verify that $\beta_{n+2}(T)/\beta_{n+1}(T) \leq \beta_{n+1}(T)/\beta_n(T)$ for all $n \geq 0$. In view of (F), this amounts to verifying $\beta_1(T)^2 \geq \beta_0(T)\beta_2(T)$, and for $n \geq 1$,

$$\left[1 + \gamma_n + \int_{[0,1]} x^n\,d\mu(x)\right]^2 \geq [1+\gamma_n]\left[1 + \gamma_n + \int_{[0,1]}(x^n + x^{n+1})\,d\mu(x)\right],$$

where $\gamma_n = \int_{[0,1]}(1 + \cdots + x^{n-1})\,d\mu(x)$. The verification is straightforward and is left to the reader. Finally, note that $n \to \varphi(n) = \Delta\beta_n(T) = \beta_{n+1}(T) - \beta_n(T)$ is completely monotone on $\mathbb{N}$. Let μ be as in Proposition 1. If $\mu([0,1]) = 0, \varphi(n) = 0$ for all n, and T is simply the unilateral shift clearly satisfying the desired inequality for α_n. Otherwise, the sequence $\{\varphi(n)/\mu([0,1])\}$ can be looked upon as the sequence $\{\beta_n(S)\}$ corresponding to a subnormal weighted shift S, and as such

$$\Delta\beta_{n+2}(T)/\Delta\beta_{n+1}(T) \geq \Delta\beta_{n+1}(T)/\Delta\beta_n(T) \quad \text{for all } n \geq 0.$$

The reader can easily check that this reduces to the desired inequality for α_n. □

Corollary 1. *If $T : \{\alpha_n\}$ is a completely hyperexpansive weighted shift such that $\alpha_{n+1}^2(\alpha_{n+2}^2-1)(\alpha_n^2-1) = \alpha_n^2(\alpha_{n+1}^2-1)^2$ for some $n \geq 1$, then the equality actually holds for all $n \geq 1$.*

Proof. If T is the unilateral shift, there is nothing to prove. Otherwise, consider the subnormal shift $S : \{\alpha_n'\}$ associated with T as in the proof of Proposition 4. If the said equality holds for some $n \geq 1$, then clearly $\alpha_n' = \alpha_{n+1}'$ for that value of n. But then $\alpha_n' = \alpha_{n+1}'$ for all $n \geq 1$ in view of a result of Stampfli [St], and the desired conclusion follows. □

Note that the weight sequence $\{\alpha_n\}$ corresponding to the Dirichlet shift satisfies the equality in Corollary 1 for all $n \geq 0$. The following proposition records some spectral properties of completely hyperexpansive weighted shifts.

Proposition 5. *Let $T : \{\alpha_n\}$ be a completely hyperexpansive weighted shift. Then, for any $T \in \mathbb{C}$ with $|\lambda| < 1$, $\lambda \in \sigma_p(T^*)$, the point spectrum of T^*; and $\dim(\ker(T^* - \lambda I)) = 1$. The spectrum $\sigma(T)$ of T is the closed unit disk $\mathbb{D}$ in $\mathbb{C}$ centered at 0, and the essential spectrum $\sigma_e(T)$ of T is the unit circle $\mathbb{T}$ in $\mathbb{C}$ centered at 0. Moreover, if $|\lambda| < 1$, then $T - \lambda I$ is a Fredholm operator with the Fredholm index -1.*

Proof. Let $T : \{\alpha_n\}$ be completely hyperexpansive. If $|\lambda| < 1$, then, for $n \geq 1$, we let $x_n = (\lambda^n/\sqrt{\beta_n})x_0$, where x_0 is some non-zero complex number. Now

$$\sum_{n\geq 0} |x_n|^2 = |x_0|^2 + |x_0|^2 \sum_{n\geq 1} |\lambda|^{2n}/\beta_n \leq |x_0|^2 + |x_0|^2 \sum_{n\geq 1} |\lambda|^{2n}$$

in view of Proposition 4, so that $x = \sum_{n\geq 0} x_n e_n$ is an element of $\mathbb{H}$. It is easy to check that $T^*x = \lambda x$. Thus $\lambda \in \sigma_p(T^*)$, and this also shows that $\mathbb{D}$ is contained in $\sigma(T)$. Further, $T^*x = \lambda x$ must force the choice of x_n as above, so that $\ker(T^* - \lambda I)$ is one-dimensional. If λ is in the approximate point spectrum of T, then $|\lambda| \geq \inf\{\|Tx\| : \|x\| = 1\}$. But $\|Tx\| \geq \|x\|$ for all x, so that $|\lambda| \geq 1$. Thus $\operatorname{Ran}(T - \lambda I)$ is closed for any λ such that $|\lambda| < 1$. Since $\operatorname{Ker}(T - \lambda I)$ is clearly $\{0\}$, the operator $T - \lambda I$ is Fredholm for $|\lambda| < 1$ and its Fredholm index is -1. That $\sigma(T) \subset \mathbb{D}$ follows from the corresponding assertion proved for a 2-hyperexpansion in [R]. We will, however, present a direct proof of this to highlight the negative definiteness of $n \to \|T^n h\|^2$ for any h in $\mathbb{H}$. (Refer to the comment at the end of Remark 2.) Observe that $n \to \|T^n h\|^2$ is a negative definite function on $\mathbb{N}$ which is bounded below. By Proposition 3.3 in Chapter 4 of [B-C-R2], one has $\|T^{n+m}h\| \leq \|T^n h\| + \|T^m h\|$ for all n, $m \geq 0$. Letting $n = m$, one has $\|T^{2n}h\| \leq 2\|T^n h\|$. Resorting to an elementary argument and using the spectral radius formula, one deduces that the spectral radius of any completely hyperexpansive operator is less than or equal to 1. Thus $\sigma(T) \subset \mathbb{D}$, and from our earlier observations $\sigma(T)$ in fact equals $\mathbb{D}$. It is now also clear that $\sigma_e(T) = \mathbb{T}$. □

An extremely interesting result in the theory of harmonic analysis on semigroups is that $\psi : \mathbb{N} \to \mathbb{R}$ is completely alternating if and only if $\varphi_t : \mathbb{N} \to \mathbb{R}$ defined by

$\varphi_t(n) = \exp(-t\psi(n))$ is completely monotone for every $t > 0$ ([B-C-R2], Ch. 4, Prop. 6.10). This result will allow us to associate with every completely hyperexpansive weighted shift a one-parameter family of contractive subnormal weighted shifts. The proof of the next proposition exploits the well-known connection between positive and negative definite 'kernels' on arbitrary sets brought about by the 'Laplace Transform' (cf. [B-C-R2], Ch. 3, Thm. 2.3).

Proposition 6. *Let $T : \{\alpha_n\}$ be a weighted shift. Then T is completely hyperexpansive if and only if $S_t : \{(t(\beta_n - 1) + 1)^{1/2}/(t(\beta_{n+1} - 1) + 1)^{1/2}\}$ is a contractive subnormal weighted shift for every $t > 0$.*

Proof. Let $T : \{\alpha_n\}$ be completely hyperexpansive. Then $n \to \beta_n$, and hence $n \to \beta_n - 1$, is a completely alternating map on $\mathbb{N}$. Thus $n \to \exp(-t(\beta_n - 1))$ is a completely monotone map on $\mathbb{N}$ for every $t > 0$. If $d\mu(S) = \exp(-s)\,ds$, it follows that $n \to \gamma_n = \int_0^\infty \exp(-st(\beta_n - 1))\,d\mu(s)$, being a continuous sum of completely monotone maps on $\mathbb{N}$, is completely monotone. But $\gamma_n = 1/(t(\beta_n - 1) + 1)$. If $S_t : \{(t(\beta_n-1)+1)^{1/2}/(t(\beta_{n+1}-1)+1)^{1/2}\}$ is the corresponding weighted shift, then it is obvious from Proposition 1 and our earlier comments that S_t is a contractive subnormal. Conversely, suppose that $S_t : \{(t(\beta_n - 1) + 1)^{1/2}/(t(\beta_{n+1} - 1) + 1)^{1/2}\}$ is a contractive subnormal for every $t > 0$. Then, for any $t > 0$, $n \to \beta_n(S_t) = (t(\beta_n - 1) + 1)^{-1}$ is a completely monotone map on $\mathbb{N}$. Hence

$$n \to \{1 - (t(\beta_n - 1) + 1)^{-1}\}/t$$

is a completely alternating map on $\mathbb{N}$ for every $t > 0$. But

$$\{1 - (t(\beta_n - 1) + 1)^{-1}\}/t = \int_0^\infty [\{1 - \exp(-st(\beta_n - 1))\}/t] \exp(-s)\,ds.$$

(We have used the fact that $\int_0^\infty \exp(-s)\,ds = 1$.) Letting t tend to zero and appealing to the Lebesgue Dominated Convergence Theorem via

$$|1 - \exp(-st(\beta_n - 1))| \leq st(\beta_n - 1),$$

one finds that the map $n \to (\beta_n - 1)\int_0^\infty s\exp(-s)\,ds = \beta_n - 1$, and hence the map $n \to \beta_n$, is completely alternating on $\mathbb{N}$. This in turn yields that $T : \{\alpha_n\}$ is completely hyperexpansive. □

Remark 4. If $T : \{\alpha_n\}$ is completely hyperexpansive, then it follows from Proposition 6 that $S_1 = \{1/\alpha_n\}$ is a contractive subnormal. We may refer to S_1 as the 'Laplace Transform' of T and let $S_1 = \mathcal{L}(T)$ denote this association. If $T = U : \{1\}$, then $\mathcal{L}(T) = U$; and if $T = D : \{\sqrt{n+2}/\sqrt{n+1}\}$, then $\mathcal{L}(T) = B : \{\sqrt{n+1}/\sqrt{n+2}\}$. (The unilateral shift is completely hyperexpansive as well as a contractive subnormal; the Dirichlet shift is completely hyperexpansive and the Bergman shift a contractive subnormal.) We point out that if $S : \{\alpha_n\}$ is a contractive subnormal, then it does not necessarily follow that $T = \{1/\alpha_n\}$ is completely hyperexpansive. For example, $n \to \beta_n(M_z^{(3)}) = 2/(n+1)(n+2)$ is completely monotone on $\mathbb{N}$, but $n \to (n+1)(n+2)/2$ is not completely alternating on $\mathbb{N}$, as can be checked easily. It is known from the Berger-Shaw Theorem [B-S] that the commutator $[T^*, T] = T^*T - TT^*$ of a subnormal weighted shift is trace-class. If $T : \{\alpha_n\}$ is completely hyperexpansive, then using $\alpha_0 \geq \alpha_n \geq 1$ $(n \geq 1)$ and considering $\mathcal{L}(T)$, it can easily be shown that $[T^*, T]$ is trace-class (that is, $\sum ||\alpha_n|^2 - |\alpha_{n-1}|^2| < \infty$). The association $T \to \mathcal{L}(T)$ also allows us another verification that $\{\alpha_n\}$ is non-increasing.

Corollary 2. *If $T : \{\alpha_n\}$ is completely hyperexpansive and $\alpha_n = \alpha_{n+1}$ for any $n \geq 1$, then $\alpha_n = 1$ for all $n \geq 1$.*

Proof. Consider $\mathcal{L}(T)$ and appeal to the result of Stampfli [St] mentioned earlier to deduce $(1/\alpha_n) = (1/\alpha_{n+1})$, and hence $\alpha_n = \alpha_{n+1}$, for all $n \geq 1$. That $\alpha_1 = 1$ follows by considering $\beta_1 - 2\beta_2 + \beta_3 \leq 0$ (that is, $\alpha_0^2(1-\alpha_1^2)^2 \leq 0$). □

As in the case of subnormals, one can raise a whole paraphernalia of questions for completely hyperexpansive operators. The multidimensional version of Proposition 2 as stated in [B-C-R2] also suggests the possibility of generalizing the previous considerations to n-tuples of operators. The author believes that the study of completely hyperexpansive operators can fruitfully supplement and be supplemented by the study of subnormals, and that the rich interplay between the theories of positive and negative definite functions can be exploited further to unfathom more interesting connections between subnormals and completely hyperexpansive operators.

Note added in proof

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