

THE MULTIPLIER OPERATORS ON THE WEIGHTED PRODUCT SPACES

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ABSTRACT. In this paper, we proved the boundedness of multiplier operators on the weighted L^p product spaces.

Let $m(\xi)$ be a function on R^n and let f be a smooth function on R^n . Suppose

$$\widehat{T_m f}(\xi) = m(\xi)\widehat{f}(\xi).$$

Then $T_m f$ is called a multiplier operator. It is well known that a multiplier operator is bounded on the weighted L^p , $1 < p < \infty$, spaces for some suitable weights if the function $m(\xi)$ satisfies Hörmander's condition

$$\int_{s \leq |\xi| \leq 2s} |\partial_\xi^\alpha m(\xi)|^2 d\xi \leq s^{n-2|\alpha|}$$

for $|\alpha| \leq [n/2] + 1$ (see [3, page 418]). The keys to proving the boundedness of multiplier operator in the weighted L^p spaces are basically

(i) the Hardy-Littlewood maximal operator is bounded by the sharp function, more precisely,

$$(1) \quad \int (Mf(x))^p W(x) dx \leq C \int (f^\#(x))^p W(x) dx$$

where $W \in A_p$, $1 < p \leq \infty$, and

$$f^\#(x) = \sup_{x \in Q} \frac{1}{|Q|} \int_Q |f(y) - f_Q| dy,$$

f_Q being the average of f over cube Q in R^n ;

(ii) an estimate,

$$(T_m f)^\#(x) \leq C(M|f|^q(x))^{1/q}$$

for some $q > 1$.

The purpose of this paper is to study the boundedness of multiplier operators on the weighted L^p product spaces.

Denote

$$\text{osc}_R f = \inf_{\substack{f_1 \\ f_2}} \left(\frac{1}{|R|} \int_R |f(x_1, x_2) - f_1(x_1) - f_2(x_2)|^2 dx_1 dx_2 \right)^{1/2}$$

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where R is any rectangle in $R^{n_1} \times R^{n_2}$ and the inf is taken over all functions f_1 and f_2 depending on the variables x_1 and x_2 respectively.

As in the “one dimensional case”, if one defines the sharp function by

$$f^\#(x) = \sup_{x \in R} \text{osc}_R f,$$

then one might expect to show that the strong maximal operator

$$M_s f(x) = \sup_{\substack{t_1 > 0 \\ t_2 > 0}} \frac{1}{t_1^{n_1} t_2^{n_2}} \int_{|y_1| \leq t_1} \int_{|y_2| \leq t_2} |f(x_1 - y_1, x_2 - y_2)| dy_1 dy_2$$

is bounded by the sharp function. Unfortunately, such an observation is not true due to Carleson’s counterexample [1]. To remove the difficulty on the sharp function in order to obtain an inequality similar to (1), R. Fefferman considered a sharp operator (see [2]) defined as follows.

Definition. Let T be an L^2 bounded linear operator. Suppose there exists an operator $T^\#$ defined on positive locally square integrable functions which is monotone, i.e.

$$T^\# f(x) \leq T^\# g(x)$$

if $f(x) \leq g(x)$ for all $x \in R^{n_1} \times R^{n_2}$ such that $\text{osc}_R(Tf) \leq \gamma^{-\sigma} T^\# f(x)$ for all $x \in R$, R a rectangle on $R^{n_1} \times R^{n_2}$, and for some $\sigma > 0$, where f is supported outside of the γ -fold dilation of R , $\gamma \geq 2$.

Based on this definition of a sharp operator, R. Fefferman [2] obtained the following inequality:

$$\int \int S^2(Tf)(x)\phi(x)dx \leq C \int \int [(I + T^\#)(|f|)]^2(x)M_s(M_s(M_s(M_s(\phi))))(x)dx$$

where I denotes the identity operator and S is the area function defined on the product spaces. Using this inequality, he obtained the following theorem, which we will apply in this paper.

Theorem A ([2]). *If T is a bounded linear operator on $L^2(R^{n_1} \times R^{n_2})$ whose sharp operator is*

$$T^\# f = M_s(f^2)^{1/2},$$

then for $p > 2$

$$\int_{R^{n_1} \times R^{n_2}} |Tf|^p W \leq C \int_{R^{n_1} \times R^{n_2}} |f|^p W$$

whenever $W \in A_{p/2}(R^{n_1} \times R^{n_2})$.

Proof. See [2, page 123]. □

In this paper, we will prove the following Theorem.

Theorem. Let $m(\xi_1, \xi_2)$ be a function and $m \in C^{p_1}(R^{n_1} \setminus \{0\}) \times C^{p_2}(R^{n_2} \setminus \{0\})$, where $p_1 = [n_1/2] + 1$, and $p_2 = [n_2/2] + 1$. Suppose

$$(2) \quad \int_{|\xi_1| \approx s_1} \int_{|\xi_2| \approx s_2} |\partial_{\xi_1}^{\alpha_1} \partial_{\xi_2}^{\alpha_2} m(\xi_1, \xi_2)|^2 d\xi_1 d\xi_2 \leq C s_1^{-2|\alpha_1|+n_1} s_2^{-2|\alpha_2|+n_2};$$

$$\sup_{\xi_2} \int_{|\xi_1| \approx s_1} |\partial_{\xi_1}^{\alpha_1} m(\xi_1, \xi_2)|^2 d\xi_1 d\xi_2 \leq C s_1^{-2|\alpha_1|+n_1};$$

$$\sup_{\xi_1} \int_{|\xi_2| \approx s_2} |\partial_{\xi_2}^{\alpha_2} m(\xi_1, \xi_2)|^2 d\xi_1 d\xi_2 \leq C s_2^{-2|\alpha_2|+n_2}$$

for every $|\alpha_1| \leq p_1$ and $|\alpha_2| \leq p_2$, where $|\xi_i| \approx s_i$ signifies that $s_i \leq |\xi_i| \leq 2s_i$. Then i) for $2 < p < \infty$,

$$\int |T_m f|^p W \leq C \int |f|^p W$$

whenever $W \in A_{p/2}(R^{n_1} \times R^{n_2})$; and ii) for $1 < p < 2$,

$$\int |T_m f|^p W \leq C \int |f|^p W$$

whenever $W^{\frac{2}{2-p}} \in A_{\frac{p}{2-p}}(R^{n_1} \times R^{n_2})$.

Remark. The weighted norm inequality for $p = 2$ can be obtained by using the interpolation theorem.

Proof. From Theorem A, we need to show that the sharp operator $T^\# f = M_s(f^2)^{1/2}$. Then (i) of our Theorem follows from Theorem A.

Let us take a smooth function ϕ on R^1 whose Fourier transform $\widehat{\phi}(t)$ has compact support $\{1/2 < |t| < 2\}$ such that $\sum \widehat{\phi}(2^{-j}|t|) = 1$ for all $t \neq 0$. Let

$$m_{i,j}(\xi_1, \xi_2) = m(\xi_1, \xi_2) \widehat{\phi}(2^{-i}|\xi_1|) \widehat{\phi}(2^{-j}|\xi_2|)$$

and

$$\widehat{T_{i,j} f}(\xi_1, \xi_2) = m_{i,j}(\xi_1, \xi_2) \widehat{f}(\xi_1, \xi_2) \equiv (\widehat{k_{i,j} * f})(\xi_1, \xi_2).$$

It is clear that $Tf = \sum_{i,j} T_{i,j} f$.

To prove $T^\# f(x) = (M_s f^2(x))^{1/2}$, one needs to estimate, for every rectangle R ,

$$(3) \quad \text{osc}_R(Tf)(x) \leq C \gamma^{-\delta} (M_s f^2(x))^{1/2}$$

for every $x \in R$ where f is supported outside of the γ -fold dilation of the rectangle R , $\gamma \geq 2$, i.e. $\text{supp } f \subset \widetilde{R}_\gamma^c$. By the homogeneity of multiplier operators, it suffices to assume R is the unit square. Since the estimates are translation invariant, we may assume the center of R is at the origin. Let us write a function f , $\text{supp } f \subset \widetilde{R}_\gamma^c$, as the sum of the functions $g + h + G$ where

$$\text{support of } g \subset {}^c \widetilde{R}_\gamma^1 \equiv \{|y_1| > \gamma, |y_2| \leq \gamma\};$$

$$\text{support of } h \subset {}^c \widetilde{R}_\gamma^3 \equiv \{|y_1| > \gamma, |y_2| > \gamma\};$$

$$\text{support of } G \subset {}^c \widetilde{R}_\gamma^2 \equiv \{|y_1| \leq \gamma, |y_2| > \gamma\}.$$

Without loss of generality it suffices to show (3) for a function $f = g + h$ where

$$\text{supp } g \subset {}^c\tilde{R}_\gamma^1 \subset \{|y_1| > \gamma, |y_2| \leq 2\};$$

$$\text{supp } h \subset {}^c\tilde{R}_\gamma^3 \subset \{|y_1| > \gamma, |y_2| > 2\}.$$

We are going to estimate

$$\text{osc}_R(Tg)(x) \leq C\gamma^{-\sigma}(M_s g^2(x))^{1/2} \text{ and } \text{osc}_R(Th)(x) \leq C\gamma^{-\sigma}(M_s h^2(x))^{1/2},$$

since $\text{osc}_R Tf \leq \text{osc}_R Tg + \text{osc}_R Th$. Let us write

$$\begin{aligned} \text{osc}_R(Tf) &\leq \sum_{i \geq 0} \text{osc}_R\left(\sum_j T_{i,j}g\right) + \sum_{i < 0} \text{osc}_R\left(\sum_j T_{i,j}g\right) \\ &\quad + \sum_{i \geq 0} \sum_{j \geq 0} \text{osc}_R T_{i,j}h + \sum_{i \geq 0} \sum_{j \leq 0} \text{osc}_R T_{i,j}h \\ &\quad + \sum_{i < 0} \sum_{j \geq 0} \text{osc}_R T_{i,j}h + \sum_{i < 0} \sum_{j < 0} \text{osc}_R T_{i,j}h \\ &\equiv I + II + III + IV + V + VI. \end{aligned}$$

We will estimate that the first two terms are dominated by $\gamma^{-\sigma}(M_s g^2(x))^{1/2}$ and the last four terms are dominated by $\gamma^{-\sigma}(M_s h^2(x))^{1/2}$ for every $x \in R$. Denote $\sum_j T_{i,j}g \equiv T_i g$ and write

$$\begin{aligned} I &= \sum_{i \geq 0} \text{osc}_R(T_i g) \\ &\leq \sum_{i \geq 0} \left(\frac{1}{|R|} \int_R |T_i g(x_1, x_2)|^2 dx_1 dx_2 \right)^{1/2} \\ (4) \quad &= \sum_{i \geq 0} \left(\int_{|x_1| \leq 1} \int_{|x_2| \leq 1} \left| \int_{|y_2| < 2} \int_{|y_1| \geq \gamma} K_i(x_1 - y_1, x_2 - y_2) g(y_1, y_2) \right. \right. \\ &\quad \left. \left. \cdot dy_1 dy_2 \right|^2 dx_2 dx_1 \right)^{1/2} \\ &\leq \sum_{i \geq 0} \sum_{2^{k_1} \geq \gamma/2} \left(\int_{|x_1| \leq 1} \int_{|x_2| \leq 1} \left| \int_{|y_2| < 2} \int_{|y_1| \approx 2^{k_1}} \right. \right. \\ &\quad \left. \left. \cdot K_i(x_1 - y_1, x_2 - y_2) g(y_1, y_2) dy_1 dy_2 \right|^2 dx_2 dx_1 \right)^{1/2}. \end{aligned}$$

Since

$$\int_{|y_1| \approx 2^{k_1}} \int_{|y_2| < 2} K_i(x_1 - y_1, x_2 - y_2) g(y_1, y_2) dy_2 dy_1$$

is a convolution operator in the variable y_2 , applying Plancherel's Theorem for the variable x_2 on (4), one has

$$I \leq \sum_{i \geq 0} \sum_{2^{k_1} \geq \gamma/2} \left(\int_{|x_1| \leq 1} \int_{\xi_2} \left| \int_{|y_1| \approx 2^{k_1}} \widehat{K}_i^2(x_1 - y_1, \xi_2) \widehat{g}^2(y_1, \xi_2) dy_1 \right|^2 d\xi_2 dx_1 \right)^{1/2}$$

where $\widehat{\cdot}^2$ denotes the Fourier transform on the second variable. For $|x_1| \leq 1$, $|y_1| \geq \gamma$, $\gamma \geq 2$, $|y_1| \approx 2^{k_1}$ then $|x_1 - y_1| \approx 2^{k_1}$. Hence

$$I \leq \sum_{i \geq 0} \sum_{2^{k_1} \geq \gamma/2} 2^{-k_1 p_1} \left(\int_{|x_1| \leq 1} \int_{\xi_2} \int_{|y_1| \approx 2^{k_1}} |x_1 - y_1|^{p_1} \widehat{K}_i^2(x_1 - y_1, \xi_2) \cdot \widehat{g}^2(y_1, \xi_2) dy_1 \right)^{1/2} d\xi_2 dx_1.$$

By Hölder's inequality and changing variable y_1 ,

$$\begin{aligned} I &\leq \sum_{i \geq 0} \sum_{2^{k_1} \geq \gamma/2} 2^{-k_1 p_1} \left(\int_{|x_1| \leq 1} \int_{\xi_2} \int \|x_1 - y_1\|^{p_1} \widehat{K}_i^2(x_1 - y_1, \xi_2) dy_1 \right. \\ &\quad \left. \cdot \int_{|y_1| \approx 2^{k_1}} |\widehat{g}^2(y_1, \xi_2)|^2 dy_1 d\xi_2 dx_1 \right)^{1/2} \\ &\leq \sum_{i \geq 0} \sum_{2^{k_1} \geq \gamma/2} 2^{-k_1 p_1} \left(\sup_{\xi_2} \int \|y_1\|^{p_1} \widehat{K}_i^2(y_1, \xi_2) dy_1 \right)^{1/2} \\ &\quad \cdot \left(\int \int_{|y_1| \approx 2^{k_1}} |\widehat{g}^2(y_1, \xi_2)|^2 dy_1 d\xi_2 \right)^{1/2} \\ &\leq \sum_{i \geq 0} \sum_{2^{k_1} \geq \gamma/2} 2^{k_1(-p_1+n_1/2)} \left(\sup_{\xi_2} \sum_{|\alpha_1|=p_1} \int |\partial_{\xi_1}^{\alpha_1} m_i(\xi_1, \xi_2)|^2 d\xi_1 \right)^{1/2} \\ &\quad \cdot \left(\frac{1}{2^{k_1 n_1}} \int_{|y_2| \leq 2} \int_{|y_1| \approx 2^{k_1}} |g(y_1, y_2)|^2 dy_1 dy_2 \right)^{1/2} \end{aligned}$$

where the last inequality is obtained by applying Plancherel's Theorem to both integrals and the support of g is contained by $\{|y_1| > \gamma, |y_2| \leq 2\}$. Hence, by the hypothesis (2) and $-p_1 + n_1/2 < 0$,

$$I \leq C \sum_{i \geq 0} \gamma^{-\sigma} (M_s g^2(0))^{1/2} 2^{i(-p_1+n_1/2)} \leq C \gamma^{-\sigma} (M_s g^2(0))^{1/2}.$$

For estimating II, we write

$$\begin{aligned} II &= \sum_{i < 0} \text{osc}_R \left(\sum_j T_{i,j} g \right) = \sum_{i < 0} \text{osc}_R (T_i g) \\ &\leq \sum_{i < 0} \left(\frac{1}{|R|} \int_R |T_i g(x_1, x_2) - T_i g(0, x_2)|^2 dx_1 dx_2 \right)^{1/2} \\ &= C \sum_{i < 0} \left(\int_R \left| \int_{e\widehat{R}_\gamma^1} (K_i(x_1 - y_1, x_2 - y_2) - K_i(0 - y_1, x_2 - y_2)) \right. \right. \\ &\quad \left. \left. \cdot g(y_1, y_2) dy_1 dy_2 \right|^2 dx_2 dx_1 \right)^{1/2} \\ &\leq C \sum_{i < 0} \sum_{2^{k_1} \geq \gamma/2} \left(\int_R \left| \int_{|y_1| \approx 2^{k_1}} \int_{|y_2| \leq 2} (K_i(x_1 - y_1, x_2 - y_2) \right. \right. \\ &\quad \left. \left. - K_i(0 - y_1, x_2 - y_2)) g(y_1, y_2) dy_1 dy_2 \right|^2 dx_2 dx_1 \right)^{1/2}. \end{aligned}$$

Here, we follow the same procedures as we did in proving I, applying Plancherel's Theorem for the variable x_2 ,

$$\begin{aligned}
 II &\leq C \sum_{i < 0} \sum_{2^{k_1} \geq \gamma/2} \left(\int_{|x_1| \leq 1} \int_{\xi_2} \left| \int_{|y_1| \approx 2^{k_1}} (\widehat{K}_i^2(x_1 - y_1, \xi_2) - \widehat{K}_i^2(0 - y_1, \xi_2)) \right. \right. \\
 &\qquad \qquad \qquad \left. \left. \cdot \widehat{g}^2(y_1, \xi_2) dy_1 \right|^2 d\xi_2 dx_1 \right)^{1/2} \\
 &= C \sum_{i < 0} \sum_{2^{k_1} \geq \gamma/2} \left(\int_{|x_1| \leq 1} \int_{\xi_2} \left| \int_{|y_1| \approx 2^{k_1}} \int_0^1 x_1 \partial_{y_1} \widehat{K}_i^2(x_1 s_1 - y_1, \xi_2) \right. \right. \\
 &\qquad \qquad \qquad \left. \left. \cdot \widehat{g}^2(y_1, \xi_2) ds_1 dy_1 \right|^2 d\xi_2 dx_1 \right)^{1/2} \\
 &\leq C \sum_{i < 0} \sum_{2^{k_1} \geq \gamma/2} \left(\int_0^1 \int_{|x_1| \leq 1} \int_{\xi_2} \left(\int_{|y_1| \approx 2^{k_1}} |\partial_{y_1} \widehat{K}_i^2(x_1 s_1 - y_1, \xi_2)|^2 dy_1 \right) \right. \\
 &\qquad \qquad \qquad \left. \cdot \left(\int_{|y_1| \approx 2^{k_1}} |\widehat{g}^2(y_1, \xi_2)|^2 dy_1 \right) d\xi_2 dx_1 ds_1 \right)^{1/2} \\
 &\leq C \sum_{i < 0} \sum_{2^{k_1} \geq \gamma/2} 2^{k_1(-p_1 + \epsilon_1 + n_1/2)} \left[\int_0^1 \int_{|x_1| \leq 1} \sup_{\xi_2} \left(\int_{|y_1| \approx 2^{k_1}} \|x_1 s_1 - y_1\|^{p_1 - \epsilon_1} \right. \right. \\
 &\qquad \qquad \qquad \left. \left. \cdot \partial_{y_1} \widehat{K}_i^2(x_1 s_1 - y_1, \xi_2) \right|^2 dy_1 \right) \right. \\
 &\qquad \qquad \qquad \left. \cdot \left(\frac{1}{2^{k_1 n_1}} \int \int_{|y_1| \approx 2^{k_1}} |\widehat{g}^2(y_1, \xi_2)|^2 dy_1 d\xi_2 \right) dx_1 ds_1 \right]^{1/2}.
 \end{aligned}$$

Taking a very small $\epsilon_1 > 0$ such that $-p_1 + \epsilon_1 + n_1/2 < 0$, changing variable (i.e. $x_1 s_1 - y_1 \rightarrow y_1$) in the integral in the first parentheses and applying Plancherel's Theorem for the integral in the second parentheses, one has

$$\begin{aligned}
 II &\leq C \gamma^{-\sigma} \sum_{i < 0} \sup_{\xi_2} \left(\int \|y_1\|^{-\epsilon_1} |y_1|^{p_1} \partial_{y_1} \widehat{K}_i^2(y_1, \xi_2) \right)^{1/2} (M_s g^2(0))^{1/2} \\
 &\leq C \gamma^{-\sigma} (M_s g^2(0))^{1/2} \sum_{i < 0} \sup_{\xi_2} \sum_{|\alpha_1|=p_1} \left(\int \|\xi_1\|^{-n_1 + \epsilon_1} *^1 \partial_{\xi_1}^{\alpha_1} (\xi_1 m_i(\xi_1, \xi_2)) \right)^{1/2} d\xi_1
 \end{aligned}$$

for some $\sigma > 0$, where $*^1$ is the convolution operator on the first variable. By fractional integration,

$$II \leq C \gamma^{-\sigma} (M_s g^2(0))^{1/2} \sum_{i < 0} \sum_{|\alpha_1|=p_1} \sup_{\xi_2} \left(\int |\partial_{\xi_1}^{\alpha_1} (\xi_1 m_i(\xi_1, \xi_2))|^q d\xi_1 \right)^{1/q}$$

where $1/q = 1/2 + \epsilon_1/n_1$ (clearly $q < 2$). By Hölder's inequality,

$$\begin{aligned} II &\leq C\gamma^{-\sigma}(M_s g^2(0))^{1/2} \sum_{i < 0} \sum_{|\alpha_1|=p_1} 2^{in_1(1/q-1/2)} \sup_{\xi_2} \left(\int |\partial_{\xi_1}^{\alpha_1}(\xi_1 m_i(\xi_1, \xi_2))|^2 d\xi_1 \right)^{1/2} \\ &\leq C\gamma^{-\sigma}(M_s g^2(0))^{1/2} \sum_{i < 0} 2^{i(-p_1+1+n_1/q)} \\ &\leq C\gamma^{-\sigma}(M_s g^2(0))^{1/2} \end{aligned}$$

(since $\epsilon_1 > 0$ then $-p_1 + 1 + n_1/q > 0$).

For estimating III, we write

$$\begin{aligned} &|T_{i,j}h(x_1, x_2)| \\ &\leq \sum_{k_2 \geq 1} \sum_{2^{k_1} \geq \gamma/2} \int_{|y_1| \approx 2^{k_1}} \int_{|y_2| \approx 2^{k_2}} |K_{i,j}(x_1 - y_1, x_2 - y_2)h(y_1, y_2)| dy_1 dy_2 \\ &\leq \sum_{k_2 \geq 1} \sum_{2^{k_1} \geq \gamma/2} \left(2^{k_1 n_1} 2^{k_2 n_2} \int_{|y_1| \approx 2^{k_1}} \int_{|y_2| \approx 2^{k_2}} |K_{i,j}(x_1 - y_1, x_2 - y_2)|^2 dy_1 dy_2 \right)^{1/2} \\ &\quad \cdot \left(\frac{1}{2^{k_1 n_1} 2^{k_2 n_2}} \int_{|y_1| \approx 2^{k_1}} \int_{|y_2| \approx 2^{k_2}} |h|^2 dy_1 dy_2 \right)^{1/2} \\ &\leq C(M_s h^2(0))^{1/2} \sum_{k_2 \geq 1} \sum_{2^{k_1} \geq \gamma/2} 2^{k_1(-p_1+n_1/2)} 2^{k_2(-p_2+n_2/2)} \\ &\quad \cdot \left(\iint ||x_1 - y_1|^{p_1} |x_2 - y_2|^{p_2} K_{i,j}(x_1 - y_1, x_2 - y_2)|^2 dy_1 dy_2 \right)^{1/2}. \end{aligned}$$

As before, applying a change of variable and Plancherel's Theorem,

$$\begin{aligned} &|T_{i,j}h(x_1, x_2)| \\ &\leq C\gamma^{-\sigma}(M_s h^2(0))^{1/2} \sum_{|\alpha_1|=p_1} \sum_{|\alpha_2|=p_2} \left(\iint |\partial_{\xi_1}^{\alpha_1} \partial_{\xi_2}^{\alpha_2} m_{i,j}(\xi_1, \xi_2)|^2 d\xi_1 d\xi_2 \right)^{1/2} \\ &\leq C\gamma^{-\sigma}(M_s h^2(0))^{1/2} 2^{i(-p_1+n_1/2)} 2^{j(-p_2+n_2/2)}. \end{aligned}$$

Hence

$$III = \sum_{i \geq 0} \sum_{j \geq 0} \text{osc}_R T_{i,j}h \leq C\gamma^{-\sigma}(M_s h^2(0))^{1/2}.$$

Since the estimates for IV and V are similar, we estimate only term V. First let us write

$$\begin{aligned}
 & |T_{i,j}h(x_1, x_2) - T_{i,j}h(0, x_2)| \\
 & \leq \sum_{k_2 \geq 1} \sum_{2^{k_1} \geq \gamma/2} \int_{|y_1| \approx 2^{k_1}} \int_{|y_2| \approx 2^{k_2}} |K_{i,j}(x_1 - y_1, x_2 - y_2) - K_{i,j}(0 - y_1, x_2 - y_2)| \\
 & \qquad \qquad \qquad \cdot |h(y_1, y_2)| dy_1 dy_2 \\
 & \leq \sum_{k_2 \geq 1} \sum_{2^{k_1} \geq \gamma/2} 2^{k_1 n_1/2} 2^{k_2 n_2/2} \left(\int_{|y_1| \approx 2^{k_1}} \int_{|y_2| \approx 2^{k_2}} |K_{i,j}(x_1 - y_1, x_2 - y_2) \right. \\
 & \qquad \qquad \qquad \left. - K_{i,j}(0 - y_1, x_2 - y_2)|^2 dy_1 dy_2 \right)^{1/2} (M_s h^2(0))^{1/2} \\
 & \leq \sum_{k_2 \geq 1} \sum_{2^{k_1} \geq \gamma/2} 2^{k_1 n_1/2} 2^{k_2 n_2/2} (M_s h^2(0))^{1/2} \left(\int_0^1 \int \int |\partial_{y_1} K_{i,j}(x_1 s_1 - y_1, x_2 - y_2)|^2 \right. \\
 & \qquad \qquad \qquad \left. \cdot dy_1 dy_2 ds \right)^{1/2} \\
 & \approx \sum_{k_2 \geq 1} \sum_{2^{k_1} \geq \gamma/2} 2^{k_1(-p_1 + \epsilon_1 + n_1/2)} 2^{k_2(-p_2 + n_2/2)} (M_s h^2(0))^{1/2} \left(\int_0^1 \int \int \right. \\
 & \qquad \qquad \qquad \left. \cdot |x_1 s_1 - y_1|^{p_1 - \epsilon_1} |x_2 - y_2|^{p_2} \partial_{y_1} K_{i,j}(x_1 s_1 - y_1, x_2 - y_2)|^2 dy_1 dy_2 ds \right)^{1/2}.
 \end{aligned}$$

Taking a positive small ϵ_1 , changing variables, applying Plancherel's Theorem and fractional integration, we get

$$\begin{aligned}
 & |T_{i,j}h(x_1, x_2) - T_{i,j}h(0, x_2)| \\
 & \leq C \gamma^{-\sigma} (M_s h^2(0))^{1/2} \\
 & \qquad \cdot \sum_{|\alpha_1|=p_1} \sum_{|\alpha_2|=p_2} \left(\int \int \|\xi_1\|^{-n_1 + \epsilon_1} *^1 \partial_{\xi_1}^{\alpha_1} \partial_{\xi_2}^{\alpha_2} (\xi_1 m_{i,j}(\xi_1, \xi_2)) \|^2 d\xi_1 d\xi_2 \right)^{1/2} \\
 & \leq C \gamma^{-\sigma} (M_s h^2(0))^{1/2} \\
 & \qquad \cdot \sum_{|\alpha_1|=p_1} \sum_{|\alpha_2|=p_2} \left(\int \left(\int |\partial_{\xi_1}^{\alpha_1} \partial_{\xi_2}^{\alpha_2} |(\xi_1 m_{i,j}(\xi_1, \xi_2))|^q d\xi_1 \right)^{2/q} d\xi_2 \right)^{1/2}
 \end{aligned}$$

where $1/q = 1/2 + \epsilon_1/n_1$. By Hölder's inequality and the hypothesis (2),

$$\begin{aligned}
 & |T_{i,j}h(x_1, x_2) - T_{i,j}h(0, x_2)| \\
 & \leq C \gamma^{-\sigma} (M_s h^2(0))^{1/2} 2^{i n_1 (1/q - 1/2)} \left(\int \int |\partial_{\xi_1}^{\alpha_1} \partial_{\xi_2}^{\alpha_2} (\xi_1 m_{i,j}(\xi_1, \xi_2))|^2 d\xi_1 d\xi_2 \right)^{1/2} \\
 & \leq C \gamma^{-\sigma} (M_s h^2(0))^{1/2} 2^{i(-p_1 + 1 + n_1/q)} 2^{j(-p_2 + n_2/2)}.
 \end{aligned}$$

Since $-p_1 + 1 + n_1/q > 0$ for positive small ϵ_1 and $-p_2 + n_2/2 < 0$, we have

$$V = \sum_{i < 0} \sum_{j \geq 0} \text{osc}_R T_{i,j}h \leq C \gamma^{-\sigma} (M_s h^2(0))^{1/2}.$$

For the last term VI, we consider

$$\begin{aligned}
 & |T_{i,j}h(x_1, x_2) - T_{i,j}h(0, x_2) - T_{i,j}h(x_1, 0) + T_{i,j}h(0, 0)| \\
 & \leq \sum_{k_2 \geq 1} \sum_{2^{k_1} \geq \gamma/2} \int_{|y_1| \approx 2^{k_1}} \int_{|y_2| \approx 2^{k_2}} (K_{i,j}(x_1 - y_1, x_2 - y_2) - K_{i,j}(-y_1, x_2 - y_2) \\
 & \quad - K_{i,j}(x_2 - y_1, -y_2) + K_{i,j}(-y_1, -y_2))h(y_1, y_2)dy_1dy_2 \\
 & \leq \sum_{k_2 \geq 1} \sum_{2^{k_1} \geq \gamma/2} \left(2^{k_1 n_1} 2^{k_2 n_2} \int_{|y_1| \approx 2^{k_1}} \int_{|y_2| \approx 2^{k_2}} |K_{i,j}(x_1 - y_1, x_2 - y_2) \right. \\
 & \quad \left. - K_{i,j}(-y_1, x_2 - y_2) - K_{i,j}(x_2 - y_1, -y_2) + K_{i,j}(-y_1, -y_2)|^2 dy_1 dy_2 \right)^{1/2} \\
 & \quad \cdot \left(\frac{1}{2^{k_1 n_1} 2^{k_2 n_2}} \int_{|y_1| \approx 2^{k_1}} \int_{|y_2| \approx 2^{k_2}} |h|^2 dy_1 dy_2 \right)^{1/2}.
 \end{aligned}$$

By the Taylor formula and Hölder's inequality,

$$\begin{aligned}
 & |T_{i,j}h(x_1, x_2) - T_{i,j}h(0, x_2) - T_{i,j}h(x_1, 0) + T_{i,j}h(0, 0)| \\
 & \leq \sum_{k_2 \geq 1} \sum_{2^{k_1} \geq \gamma/2} (M_s h^2(0))^{1/2} \left(2^{k_1 n_1} 2^{k_2 n_2} \int_0^1 \int_0^1 \int_{|y_1| \approx 2^{k_1}} \int_{|y_2| \approx 2^{k_2}} \right. \\
 & \quad \left. \cdot |\partial_{y_1} \partial_{y_2} K_{i,j}(x_1 s_1 - y_1, x_2 s_2 - y_2)|^2 dy_1 dy_2 ds_1 ds_2 \right)^{1/2} \\
 & \approx C(M_s h^2(0))^{1/2} \sum_{k_2 \geq 1} \sum_{2^{k_1} \geq \gamma/2} 2^{k_1(-p_1 + \epsilon_1 + n_1/2)} 2^{k_2(-p_2 + \epsilon_2 + n_2/2)} \\
 & \quad \cdot \left(\int_0^1 \int_0^1 \int_{|y_1| \approx 2^{k_1}} \int_{|y_2| \approx 2^{k_2}} ||x_1 s_1 - y_1|^{p_1 - \epsilon_1} |x_2 s_2 - y_2|^{p_2 - \epsilon_2} \right. \\
 & \quad \left. \cdot \partial_{y_1} \partial_{y_2} K_{i,j}(x_1 s_1 - y_1, x_2 s_2 - y_2)|^2 dy_1 dy_2 ds_1 ds_2 \right)^{1/2} \\
 & \leq C\gamma^{-\sigma} (M_s h^2(0))^{1/2} \int \int ||y_1|^{-\epsilon_1} |y_2|^{-\epsilon_2} |y_1|^{p_1} |y_2|^{p_2} \\
 & \quad \cdot \partial_{y_1} \partial_{y_2} K_{i,j}(y_1, y_2)|^2 dy_1 dy_2 \Big)^{1/2} \\
 & \leq C\gamma^{-\sigma} (M_s h^2(0))^{1/2} \sum_{|\alpha_1|=p_1} \sum_{|\alpha_2|=p_2} \left(\int ||\xi_1|^{-n_1 + \epsilon_1} |\xi_2|^{-n_2 + \epsilon_2} \right. \\
 & \quad \left. * \partial_{\xi_1}^{\alpha_1} \partial_{\xi_2}^{\alpha_2} (\xi_1 \xi_2 m_{i,j}(\xi_1, \xi_2))|^2 d\xi_1 d\xi_2 \right)^{1/2}.
 \end{aligned}$$

Next, we use fractional integration twice on the variables ξ_1 and ξ_2 respectively. Let $1/q_1 = 1/2 + \epsilon_1/n_1$ and $1/q_2 = 1/2 + \epsilon_2/n_2$. Then

$$\begin{aligned} & |T_{i,j}h(x_1, x_2) - T_{i,j}h(0, x_2) - T_{i,j}h(x_1, 0) - T_{i,j}h(0, 0)| \\ & \leq C\gamma^{-\sigma}(M_s h^2(0))^{1/2} \sum_{|\alpha_1|=p_1} \sum_{|\alpha_2|=p_2} \left(\int \left(\int \|\xi_2\|^{-n_2+\epsilon_2} \right. \right. \\ & \quad \left. \left. *^2 \partial_{\xi_1}^{\alpha_1} \partial_{\xi_2}^{\alpha_2} (\xi_1 \xi_2 m_{i,j}(\xi_1, \xi_2)) \right)^{q_1} d\xi_1 \right)^{2/q_1} d\xi_2 \end{aligned}$$

where $*^2$ denotes the convolution operator on the second variable. By Minkowski's inequality, the last inequality is less than

$$\begin{aligned} & C\gamma^{-\sigma}(M_s h^2(0))^{1/2} \sum_{|\alpha_1|=p_1} \sum_{|\alpha_2|=p_2} \left(\int \left(\int \|\xi_2\|^{-n_2+\epsilon_2} \right. \right. \\ & \quad \left. \left. *^2 \partial_{\xi_1}^{\alpha_1} \partial_{\xi_2}^{\alpha_2} (\xi_1 \xi_2 m_{i,j}(\xi_1, \xi_2)) \right)^{q_1/2} d\xi_1 \right)^{1/q_1} \\ & \leq C\gamma^{-\sigma}(M_s h^2(0))^{1/2} \sum_{|\alpha_1|=p_1} \sum_{|\alpha_2|=p_2} \left(\int \left(\int |\partial_{\xi_1}^{\alpha_1} \partial_{\xi_2}^{\alpha_2} (\xi_1 \xi_2 m_{i,j}(\xi_1, \xi_2))|^{q_2} \right. \right. \\ & \quad \left. \left. d\xi_2 \right)^{q_1/q_2} d\xi_1 \right)^{1/q_1} \\ & \leq C\gamma^{-\sigma}(M_s h^2(0))^{1/2} \sum_{|\alpha_1|=p_1} \sum_{|\alpha_2|=p_2} 2^{jn_2(1/q_2-1/2)} \\ & \quad \cdot \left(\int \left(\int |\partial_{\xi_1}^{\alpha_1} \partial_{\xi_2}^{\alpha_2} (\xi_1 \xi_2 m_{i,j}(\xi_1, \xi_2))|^{q_2} d\xi_2 \right)^{q_1/2} d\xi_1 \right)^{2/q_1} \\ & \leq C\gamma^{-\sigma}(M_s h^2(0))^{1/2} \sum_{|\alpha_1|=p_1} \sum_{|\alpha_2|=p_2} 2^{jn_2(1/q_2-1/2)} 2^{in_1(1/q_1-1/2)} \\ & \quad \cdot \left(\int \int \partial_{\xi_1}^{\alpha_1} \partial_{\xi_2}^{\alpha_2} |(\xi_1 \xi_2 m_{i,j}(\xi_1, \xi_2))|^{q_2} d\xi_1 d\xi_2 \right)^{1/2} \\ & \leq C\gamma^{-\sigma}(M_s h^2(0))^{1/2} 2^{(-p_1+1+n_1/q_1)} 2^{(-p_2+1+n_2/q_2)}. \end{aligned}$$

Hence

$$VI \leq \sum_{i < 0} \sum_{j < 0} \text{osc}_R T_{i,j}h \leq C\gamma^{-\sigma}(M_s h^2(0))^{1/2}.$$

Combining the above estimates, we conclude that

$$T^\# f(x_1, x_2) = (M_s f^2(x))^{1/2}.$$

(i) is proved.

For the proof of (ii), we use duality. Let $U = W^{-1/(p-1)}$. Then the dual space of $L_W^p(R^{n_1} \times R^{n_2})$ is $L_U^{p'}(R^{n_1} \times R^{n_2})$. There exists a function $g \in L_U^{p'}(R^{n_1} \times R^{n_2})$ such that

$$\|T_m f\|_{L_W^p} = \int T_m f \bar{g} = \int \overline{f T_m g} \leq \|f\|_{L_W^p} \|T_m g\|_{L_U^{p'}}.$$

It is easy to see that

$$W^{2/(2-p)} \in A_{p/(2-p)} \iff U \in A_{p'/2}.$$

Applying (i),

$$\|T_m g\|_{L_U^{p'}} \leq C \|g\|_{L_U^{p'}}$$

where $U \in A_{p'/2}$. The Theorem is proved. \square

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