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# ON THE PLURICANONICAL MAP OF THREEFOLDS OF GENERAL TYPE

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ABSTRACT. Let $X$ be a smooth minimal threefold of general type and let $n$ be an integer $> 1$. Assume that the image of the pluricanonical map $\Phi_n$ of $X$ is a curve. Then a simple computation shows that $n$ is necessarily 2 or 3. When $n = 2$ with a numerical condition or when $n = 3$, we obtain two inequalities $\chi(\mathcal{O}_X) \le \min\{-1, 2 - 2q_1\}$ and $q_1 \le \frac{3}{14}{K_X}^3 + 1$, where $q_1$ is the irregularity of $X$ and $\chi(\mathcal{O}_X)$ is the Euler characteristic of $X$.

Throughout this paper, we are working over the complex number field **C**.

In this paper, we have studied the case that the image $C$ of the pluricanonical map $\Phi_n$ is a curve for an integer $n > 1$. In this case, a simple calculation shows $n = 2$ or 3. Resolve the base locus of $\Phi_n$. Then we have two terms '$b$' and '$c$' explained just below Proposition 1. When $n = 2$ with additional numerical conditions or when $n = 3$, we may have some information about $\Phi_n$ from these two terms, which are explained in Corollary of Proposition 2 and Proposition 3. Using this information, we obtain two inequalities $\chi(\mathcal{O}_X) \le \min\{-1, 2 - 2q_1\}$ and $q_1 \le \frac{3}{14}{K_X}^3 + 1$, which are given in Theorems 2 and 3.

Now, let's set up our notations. Let $X$ be a smooth projective variety and let $D$ be a divisor on $X$. Denote by $K_X$ the canonical divisor of $X$. Denote by $\Phi_n$ the rational map associated to the complete linear system $|nK_X|$. Denote by $h^i(X, \mathcal{O}_X(D))$ the dimension of $H^i(X, \mathcal{O}_X(D))$. Let $\mathrm{Bs}|D|$ mean the base locus of $|D|$. Let's denote the genus of $X$ by $p_g(X)$ and $h^i(X, \mathcal{O}_X)$ by $q_i(X)$. (or simply $p_g$ and $q_i$ unless there is any confusion.) Denote by $\sim$ the linear equivalence and by $\equiv$ the numerical equivalence. For a real number $r$, $[r]$ means the greatest among the integers less than or equal to $r$.

**Theorem 1** (Kawamata-Viehweg vanishing theorem)**.** *Let $X$ be a nonsingular projective variety. If $D$ is a nef and big divisor on $X$, then $H^i(X, \mathcal{O}_X(K_X + D)) = 0$ for all $i > 0$.*

For a reference, see KMM [5].

**Lemma.** *Let $X$ be a smooth projective threefold, and let $D$ be a divisor on $X$. Then we have the following:*

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(a) $\chi(\mathcal{O}_X(D)) = D^3/6 \;-\; K_X\cdot D^2/4 \;+\; D\cdot({K_X}^2+c_2)/12 \;+\; \chi(\mathcal{O}_X)$, *where* $c_2$ *is the second Chern class of* $X$. *Moreover* $\chi(\mathcal{O}_X) = -c_2\cdot K_X/24$.

(b) $K_X\cdot D^2$ *is even.*

(c) $p_n :\stackrel{def}{=} h^0(X,\mathcal{O}_X(nK_X)) = \dfrac{n(n-1)(2n-1)}{12}{K_X}^3+(1-2n)\chi(\mathcal{O}_X)$ *for* $n\geq 2$ *and* $\chi(\mathcal{O}_X)<0$ *when* $K_X$ *is nef and big.*

*Proof.* (a) comes from the Riemann-Roch theorem.

(b) comes from the following:

$$\chi(\mathcal{O}_X(D))+\chi(\mathcal{O}_X(-D)) = -K_X\cdot D^2/2+2\chi(\mathcal{O}_X)\in\mathbf{Z}.$$

(c) comes from (a) and the Kawamata-Viehweg vanishing theorem. We have $\chi(\mathcal{O}_X)<0$ since $-c_2\cdot K_X\leq -{K_X}^3/3$. (See Miyaoka [8].) □

**Proposition 1.** *Let* $C$ *be a nondegenerate curve of degree* $a$ *in* $\mathbf{P}^n$. *Then we have*

(a) *If* $n\leq a<2n$, *then* $p_g(C)\leq a-n$.

(b) *If* $2n\leq a$, *then* $p_g(C)\leq \dfrac{m(m-1)}{2}(n-1)+mr$, *where* $m=\left[\dfrac{a-1}{n-1}\right]$ *and* $a-1=m(n-1)+r$.

*Proof.* See Griffiths & Harris [2], p. 253. □

We are going to use the following notations in the rest of this paper.

Let $X$ be a smooth projective threefold of general type with $K_X$ nef. Let $n$ be a positive integer $\geq 2$.

Suppose that the dimension of the image $C$ of $\Phi_n$ is 1.

Let $|nK_X| = |M| + Z$, where $|M|$ and $Z$ are the moving part and fixed part of $|nK_X|$ respectively. Let $f : X'\to X$ be the resolution of the base locus of $\Phi_n$ if $\mathrm{Bs}|nK_X|\neq\emptyset$, i.e., $f$ is a succession of blow-ups along nonsingular centers of codim$\geq 2$ in the base locus such that $g=\Phi_n\circ f$ is a morphism. Let $g=k\circ h$ be the Stein factorization. Observe that $C'$ is normal and hence smooth.

$$\begin{array}{ccc} X' & \xrightarrow{\;h\;} & C' \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle k} \\ X & \xrightarrow[\Phi_n]{} & C \end{array}$$

Let $a$ be deg$C$ in $\mathbf{P}^{p_n-1}$, and let $b=$deg$k$. Recall that $C$ is a nondegenerate curve in $\mathbf{P}^{p_n-1}$. Let $S$ be the general fiber of $h$.

We have $K_{X'} = f^*(K_X)+E'$ and $|f^*nK_X| = |M'|+Z'$, where $E'$ is the ramification divisor of $f$ supported on the exceptional locus of $f$ and $Z'$ is the fixed part of $|f^*nK_X|$. We have $M'\equiv abS$. Since $f^*K_X$ is nef, $f^*{K_X}^2\cdot Z'\geq 0$. Hence

$$n{K_X}^3 = nf^*{K_X}^3 = (abS+Z')f^*{K_X}^2 \geq abf^*{K_X}^2\cdot S.$$

Let $c=f^*{K_X}^2\cdot S$. Since $f^*K_X$ is nef and big, and $S$ is nef and not numerically equivalent to 0, we have $f^*{K_X}^2\cdot S\geq 1$. Hence

$$\frac{n{K_X}^3}{bc}\geq a\geq p_n-1. \tag{1}$$

From the inequality (1) and Lemma, the image of $\Phi_n$ can be a curve only when $n$ is 2 or 3. Moreover, we have $1\leq bc\leq 3$ and in particular, $bc=1$ when $n=3$.

**Proposition 2** (cf. Matsuki [7])**.** *If* $\dim\ \mathrm{Im}\,\Phi_n = 1$*, then* $S$ *is a surface of general type and* $S'$ *has* ${K_{S'}}^2 = f^*{K_X}^2 \cdot S$*, where* $\pi : S \to S'$ *is a morphism of* $S$ *to its minimal surface* $S'$*.*

*Proof.* The easy addition formula '$\kappa(X) \leq \kappa(S) + \dim C$' implies that $S$ is the surface of general type, where $\kappa(X)$ means the Kodaira dimension of $X$. (For a reference about easy addition formula, see Ueno [9].) We have that $nK_X \equiv abD+Z$ and $f^*abD \equiv abS + E$, where $D = f_*S$ and $E$ is an effective divisor supported on the exceptional locus of $f$. So,

$$nf^*{K_X}^2 \cdot S = n{K_X}^2 \cdot D = (abD + Z) \cdot K_X \cdot D \geq abK_X \cdot D^2.$$

Since $D^2$ is an effective 1-cycle and $K_X$ is nef, $K_X \cdot D^2 \geq 0$ . If $K_X \cdot D^2 \neq 0$, then $nf^*{K_X}^2 \cdot S \geq 2a \geq 2(p_n - 1)$ since $K_X \cdot D^2$ is even. This inequality holds true only when $n = 2$, ${K_X}^3 = 2$, $\chi(\mathcal{O}_X) = -1$ and $f^*{K_X}^2 \cdot S = 3$. (Recall that $2 \leq n \leq 3$ and $1 \leq c \leq 3$.) But it does not satisfy the inequality (1). Hence $K_X \cdot D^2 = 0$. So

$$\begin{aligned} 0 = (abD)^2 \cdot K_X &= f^*abD \cdot f^*abD \cdot f^*K_X \\ &= (abS + E) \cdot f^*abD \cdot f^*K_X \\ &= abS \cdot f^*abD \cdot f^*K_X \\ &= abS \cdot (abS + E) \cdot f^*K_X \\ &= abS \cdot E \cdot f^*K_X. \end{aligned}$$

Hence we have $S \cdot E \cdot f^*K_X = 0$. Let $\{E_i\}$ be the irreducible components of $E$. Clearly $S \cdot E_i \cdot f^*K_X = 0$ for each $i$. By the way of taking $f$, supp$(E)$ =supp$(E')$. Hence we have $S \cdot E' \cdot f^*K_X = 0$, since $S \cdot E_i \cdot f^*K_X = 0$ for each $i$.

Applying the Hodge index theorem to $S$, we have that $(E'|_S)^2 = S \cdot E'^2 \leq 0$ since $(f^*K_X|_S)^2 = f^*{K_X}^2 \cdot S \geq 1$. We have $f^*K_X|_S + E'|_S \sim K_S \sim \pi^*K_{S'} + L$, where $L$ is an effective divisor supported on the exceptional locus of $\pi$. Hence the uniqueness of the Zariski decomposition implies that $f^*K_X|_S \sim \pi^*K_{S'}$. □

**Corollary.** *If* $f^*{K_X}^2 \cdot S = 1$*, then the minimal surface* $S'$ *of* $S$ *has* ${K_{S'}}^2 = 1$*,* $q(S) = 0$ *and* $1 \leq p_g(S) \leq 2$*.*

*Proof.* ${K_{S'}}^2 = f^*K_X|_S{}^2 = 1$ by Proposition 2. Since ${K_{S'}}^2 = 1$, we have that $q(S) = 0$ and $p_g(S) \leq 2$. (For a reference, see Bombieri [1], p. 212.) Consider the exact sequence

$$0 \to \mathcal{O}_{X'}(K_{X'}) \to \mathcal{O}_{X'}(K_{X'} + M') \to \overset{ab}{\oplus}\mathcal{O}_S(K_S) \to 0.$$

The above exact sequence shows that $p_g(S) \neq 0$ since $M'$ is not fixed. □

We have two facts from the condition $bc = 1$. The first one is that $C'$ is birational to the image $C$ of the pluricanonical map $\Phi_n$. So we may assume that $C$ is smooth since the terms we are interested in are birational invariants. The second one is that a general fiber of $g$ is a surface of general type with its irregularity 0 from Corollary of Proposition 2.

**Proposition 3.** *Suppose that* $bc = 1$*. Then we have that* $1 \leq p_g(S) \leq 2$ *and* $\chi(\mathcal{O}_X) \leq (p_g(S) + 1)(1 - q_1)$*.*

*Proof.* We have the fiber space $g : X' \to C$ with connected fiber since $bc = 1$. By Corollary of Proposition 2, $q(S) = 0$ and $1 \le p_g(S) \le 2$. We have $R^1 g_* K_{X'} = 0$ since $q(S) = h^1(S, \mathcal{O}_S(K_S)) = 0$. By spectral sequence, we have that

$$\begin{aligned} p_g &= h^0(X, \mathcal{O}_X(K_X)) = h^0(C, g_* K_{X'}), \\ q_2 &= h^1(X, \mathcal{O}_X(K_X)) = h^1(C, g_* K_{X'}), \\ q_1 &= h^2(X, \mathcal{O}_X(K_X)) = h^0(C, R^2 g_* K_{X'}). \end{aligned}$$

Since $R^2 g_* K_{X'} = K_C$, $q_1 = p_g(C)$. (For a reference, see Kollár [6].) It is known that $g_* K_{X'/C} \stackrel{def}{=} g_*(K_{X'} \otimes g^* {K_C}^{-1})$ is semipositive and locally free of rank $p_g(S)$. So $\deg g_* K_{X'/C} \ge 0$. (See Kawamata [4].)

$$\begin{aligned} h^0(C, g_* K_{X'}) - h^1(C, g_* K_{X'}) &= \deg g_* K_{X'} + p_g(S)(1 - p_g(C)) \\ &= \deg g_* K_{X'/C} + p_g(S)(p_g(C) - 1) \\ &\ge p_g(S)(p_g(C) - 1). \end{aligned}$$

So $p_g - q_2 \ge p_g(S)(q_1 - 1)$. Hence $-\chi(\mathcal{O}_X) = p_g - q_2 + q_1 - 1 \ge (p_g(S) + 1)(q_1 - 1)$. Now we have $\chi(\mathcal{O}_X) \le (p_g(S) + 1)(1 - q_1)$. □

**Theorem 2.** *If* $\dim \mathrm{Im}\Phi_3 = 1$*, then the following hold:*

(a) $-\dfrac{1}{10}{K_X}^3 - \dfrac{1}{5} \le \chi(\mathcal{O}_X) \le \min\{-1,\ 2 - 2q_1\}$. *Moreover,* ${K_X}^3 \ge 8$.

(b) $q_1 \le \dfrac{1}{22}{K_X}^3 + 1$.

*Proof.* The fact $bc = 1$ is given just above Proposition 2. If $a \ge 2(p_3 - 1)$, then the inequality (1) implies $3{K_X}^3/2 \ge p_3 - 1$. But it is impossible since $p_3 = \dfrac{5}{2}{K_X}^3 - 5\chi(\mathcal{O}_X)$. So $bc = 1$ and $a < 2(p_3 - 1)$. Hence Lemma and Proposition 3 imply $\chi(\mathcal{O}_X) \le \min\{-1, 2 - 2q_1\}$. From the inequality (1), we have $-\dfrac{1}{10}{K_X}^3 - \dfrac{1}{5} \le \chi(\mathcal{O}_X)$. Moreover, since $\chi(\mathcal{O}_X) \le -1$, ${K_X}^3 \ge 8$.

For (b), by Proposition 1, $p_g(C) \le a - (p_3 - 1)$ since the degree $a$ of $C$ is less than $2(p_3 - 1)$. Recall that we have $q_1 = p_g(C)$ in the proof of Proposition 3. We already know $a \le 3{K_X}^3$ from (1). Hence

$$q_1 = p_g(C) \le a - (p_3 - 1) \le 3{K_X}^3 - p_3 + 1 \le 1/2{K_X}^3 + 5\chi(\mathcal{O}_X) + 1.$$

From (a), $q_1 \le 1/2{K_X}^3 + 5(2 - 2q_1) + 1$. Hence we have $q_1 \le \dfrac{1}{22}{K_X}^3 + 1$. □

**Theorem 3.** *Suppose that* $\dim \mathrm{Im}\Phi_2 = 1$. *If* $bc = 1$*, then we have the following:*

(a) $\chi(\mathcal{O}_X) \le \min\{-1,\ 2 - 2q_1\}$.

(b) $q_1 \le \dfrac{3}{14}{K_X}^3 + 1$.

*Proof.* Lemma and Proposition 3 imply (a) clearly.

For (b), we already know $p_2 - 1 \le a \le 2{K_X}^3$ from (1). Hence it is enough to consider the following two cases:

<u>Case 1.</u> $p_2 - 1 \le a < 2(p_2 - 1)$.

Since $p_2-1 \le a < 2(p_2-1)$, Proposition 1 implies that $p_g(C) \le a-(p_2-1)$. Since $bc = 1$, the proof of Proposition 3 implies that $q_1 = p_g(C)$ and $\chi(\mathcal{O}_X) \le 2 - 2q_1$.

$$\begin{aligned} q_1 = p_g(C) &\le a - (p_2 - 1) \\ &\le 2{K_X}^3 - p_2 + 1 \\ &= \frac{3}{2}{K_X}^3 + 3\chi(\mathcal{O}_X) + 1 \\ &\le \frac{3}{2}{K_X}^3 + 3(2 - 2q_1) + 1. \end{aligned}$$

Hence we have $q_1 \le \frac{3}{14}{K_X}^3 + 1$.

Case 2. $2(p_2 - 1) \le a \le 2{K_X}^3$.

Since $2(p_2 - 1) \le a \le 2{K_X}^3$, Proposition 1 implies that

$$p_g(C) \le \frac{m(m-1)}{2}(p_2 - 2) + mr,$$

where $m = \left[\frac{a-1}{p_2-2}\right]$ and $a - 1 = m(p_2 - 2) + r$.

So, let's compute $m$. Since $2(p_2 - 1) \le a \le 2{K_X}^3$, we have that

$$\frac{2(p_2-1)-1}{p_2-2} \le \frac{a-1}{p_2-2} \le \frac{2{K_X}^3-1}{p_2-2}.$$

If we modify the above inequalities, we have that

$$2 + \frac{1}{{K_X}^3/2 - 3\chi(\mathcal{O}_X) - 2} \le \frac{a-1}{p_2-2} \le 4 + \frac{12\chi(\mathcal{O}_X)+7}{{K_X}^3/2 - 3\chi(\mathcal{O}_X) - 2}.$$

Since $\chi(\mathcal{O}_X) < 0$, we have that

$$\frac{1}{{K_X}^3/2 - 3\chi(\mathcal{O}_X) - 2} > 0 \text{ and } \frac{12\chi(\mathcal{O}_X)+7}{{K_X}^3/2 - 3\chi(\mathcal{O}_X) - 2} < 0.$$

Hence we have $m = \left[\frac{a-1}{p_2-2}\right] = 2$ or $3$.

When $m = 2$, we have that

$$\begin{aligned} q_1 = p_g(C) &\le \frac{2\cdot 1}{2}(p_2 - 2) + 2r \\ &\le (p_2 - 2) + 2(a - 1 - 2(p_2 - 2)) \\ &\le 2a - 3p_2 + 4 \\ &\le 4{K_X}^3 - 3(\frac{{K_X}^3}{2} - 3\chi(\mathcal{O}_X)) + 4 \\ &\le \frac{5}{2}{K_X}^3 + 9\chi(\mathcal{O}_X) + 4 \\ &\le \frac{5}{2}{K_X}^3 + 9(2 - 2q_1) + 4. \end{aligned}$$

Hence $q_1 \le \frac{5}{38}{K_X}^3 + \frac{22}{19}$.

When $m = 3$, similarly, we have that $q_1 \le \frac{3}{37}{K_X}^3 + \frac{45}{37}$.

Therefore, combining all inequalities about $q_1$, we have $q_1 \le \frac{3}{14}{K_X}^3 + 1$. □

*Remark.* When $\dim \mathrm{Im}\Phi_2 = 1$, we assume the condition $bc = 1$. At this moment, we don't have a necessary and sufficient condition to guarantee $bc = 1$. But we have some cases which show $bc = 1$.

**Proposition 4.** *Suppose that* $\dim \mathrm{Im}\Phi_2 = 1$. *If* ${K_X}^3 < p_2 - 1$*, then we have that*
(a) $bc = 1$,
(b) $-{K_X}^3/2 - 1/3 \leq \chi(\mathcal{O}_X) < -{K_X}^3/6 - 1/3$.

*Proof.* Since ${K_X}^3 < p_2 - 1 \leq a$, we have ${K_X}^3 < p_2 - 1 \leq a \leq \dfrac{2{K_X}^3}{bc}$ from (1). Hence $bc$ must be 1.

From (1), we have $-{K_X}^3/2 - 1/3 \leq \chi(\mathcal{O}_X)$. Since ${K_X}^3 < p_2 - 1$, we have $\chi(\mathcal{O}_X) < -{K_X}^3/6 - 1/3$. Combining these two inequalities, we have

$$-{K_X}^3/2 - 1/3 \leq \chi(\mathcal{O}_X) < -{K_X}^3/6 - 1/3.$$

□

**Proposition 5.** *Suppose that* $\dim \mathrm{Im}\Phi_2 = 1$. *If* $2(p_2 - 1) \leq a$*, then we have*
(a) $bc = 1$,
(b) $-{K_X}^3/6 - 1/3 \leq \chi(\mathcal{O}_X) \leq -1$.

*Proof.* Since $2(p_2 - 1) \leq a$, we have $2(p_2 - 1) \leq a \leq 2{K_X}^3/bc$ from (1). If $bc \geq 2$, then we have $2(p_2 - 1) \leq {K_X}^3$. But, since $2(p_2 - 1) = {K_X}^3 - 6\chi(\mathcal{O}_X) - 2 > {K_X}^3$, it is impossible. Hence $bc = 1$.

From $2(p_2 - 1) \leq a \leq 2{K_X}^3$, we have $-{K_X}^3/6 - 1/3 \leq \chi(\mathcal{O}_X)$. □

*Remark.* When $\dim \mathrm{Im}\Phi_2 = 1$, from (1), we may have the following three cases: $a \leq 2{K_X}^3 < 2(p_2 - 1)$, $a \leq 2(p_2 - 1) \leq 2{K_X}^3$, and $2(p_2 - 1) \leq a \leq 2{K_X}^3$. The case we didn't cover here is the second one $p_2 - 1 \leq a \leq 2(p_2 - 1) \leq 2{K_X}^3$.

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