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AN EXTENSION OF THE VITALI-HAHN-SAKS THEOREM

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Abstract. The Vitali-Hahn-Saks theorem on the absolute continuity of the setwise limit of a sequence of bounded measures is extended to allow unbounded measures and convergence of integrals of continuous functions vanishing at infinity.

1. Introduction

A useful version of the Vitali-Hahn-Saks theorem gives conditions for a measure λ which is the setwise limit of a sequence of absolutely continuous (a.c.) measures $\{\lambda_n\}$ to be a.c. This result allows the underlying measure space $(X, \mathcal{B})$ to be arbitrary (cf. [2], p. 155), but it requires that $\{\lambda_n\}$ and λ, as well as the dominant measure, say μ, are all finite.

In this paper, we show that the same conclusion ($\lambda \ll \mu$) holds without requiring either *setwise* convergence or *finiteness* of the measures involved, but we impose in particular suitable topological conditions on X.

Moreover, we give conditions for the Radon-Nikodym derivative $d\lambda/d\mu$ to be in $L_p(X, \mathcal{B}, \mu)$ for $1 \leq p \leq \infty$.

In addition to being interesting in themselves, these results are also useful to study the existence of solutions to the Poisson Equation for a Markov kernel [3].

2. Main results

Let $(X, \mathcal{B}, \mu)$ be a σ-finite complete measure space, where X is a locally compact separable metric space, and $\mathcal{B}$ is the completion (with respect to μ) of the σ-algebra of Borel subsets of X. Let $C_0(X)$ be the space of real-valued continuous functions on X vanishing at infinity. (Concerning the convergence in (2.1), see Remark 2.3(b) at the end of this section.)

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Theorem 2.1. *Let* $\{\lambda_n\}$ *and* λ *be* σ*-finite (nonnegative) measures on* $(X, \mathcal{B})$ *such that, as* $n \to \infty$,

$$\int_X v d\lambda_n \to \int_X v d\lambda \ \forall v \in C_0(X). \tag{2.1}$$

If, in addition, every λ_n *is a.c. with respect to* μ*, then so is* λ.

Suppose that the measures λ and λ_n in Theorem 2.1 are finite. Then, by the Radon-Nikodym theorem, there are nonnegative measurable functions u_n and u such that

$$\lambda_n(B) = \int_B u_n d\mu \quad \text{and} \quad \lambda(B) = \int_B u d\mu \ \forall B \in \mathcal{B}.$$

The next theorem gives conditions for u to be in $L_p := L_p(X, \mathcal{B}, \mu),\ 1 \le p \le \infty$.

Theorem 2.2. *Fix* $1 \le p \le \infty$. *In addition to the hypotheses of Theorem 2.1, suppose that* $u_n \in L_p$ $\forall n$ *and that, for some constant* M, $\liminf_n ||u_n||_p \le M$. *Then*

$$u \text{ is in } L_p. \tag{2.2}$$

Remark 2.3. (a) If the λ_n are all finite, then in Theorem 2.1, instead of (2.1), we only need to assume that $\lim_n \int_X v d\lambda_n$ exists and is finite for all $v \in C_0(X)$. Then, from [2], Theorem VIII.8, it follows that the sequence $\{\lambda_n\}$ converges to a measure λ, as in (2.1).

(b) Endowed with the sup-norm, $C_0(X)$ is a Banach space and its dual is the space $M(X)$ of finite signed measures on X with total variation norm [2],[5]. Thus, when the λ_n are finite measures, (2.1) can be stated equivalently as: λ_n converges to λ in the weak* topology $\sigma(M(X), C_0(X))$. If λ_n and λ are probability measures and (2.1) holds, it is sometimes said that λ_n converges vaguely to λ. If $C_0(X)$ is replaced by the (larger) space $C(X)$ of all continuous and bounded functions on X, it is said that λ_n converges weakly to λ (see e.g. [1]). It follows from Lemma 3.1(b) and the Portmanteau Theorem ([1], Theorem 2.1) that, in this case, vague convergence implies weak convergence (the converse being obviously true).

(c) To verify (2.1) it suffices to consider nonnegative $v \in C_0(X)$. Indeed, since any $v \in C_0(X)$ can be written as $v = v^+ + v^-$ with $v^+(x) := \max[0, v(x)]$ and $v^-(x) := \min[0, v(x)]$, it suffices to observe that both v^+ and v^- are in $C_0(X)$. Hence, convergence in (2.1) for all $0 \le v \in C_0$ implies convergence for all $v \in C_0(X)$.

3. Proofs

The proof of Theorem 2.1 is based on the following lemma, which is well known in the case of finite measures (and bounded function h); see e.g. [2], Theorem VIII.10.

Lemma 3.1. *In the context of Theorem 2.1:*

(a) *If* h *is a nonnegative and l.s.c. (lower semicontinuous) function on* X*, then*

$$\liminf_{n\to\infty} \int_X h d\lambda_n \ge \int_X h d\lambda;$$

(b) $\liminf_{n\to\infty} \lambda_n(B) \ge \lambda(B)$ *for any open set* $B \in \mathcal{B}$.

Proof. (a) As h is nonnegative and l.s.c., there exists an increasing sequence of nonnegative continuous bounded functions v_k on X such that $v_k(x) \uparrow h(x)\ \forall x \in X$. Similarly, as each v_k is a nonnegative continuous bounded function and X is σ-compact ([4], p. 203, Theorem 21), for every k there is an increasing sequence $\{v_{kl},\ l = 1, 2, ...\}$ of nonnegative functions v_{kl} in $C_0(X)$ with $v_{kl}(x) \uparrow v_k(x)$ for all $x \in X$ as $l \to \infty$. Hence,

$$\begin{aligned} \liminf_n \int h d\lambda_n &\geq \liminf_n \int v_k d\lambda_n\ \forall k \\ &\geq \liminf_n \int v_{kl} d\lambda_n\ \forall k, l \\ &= \int v_{kl} d\lambda \quad \text{(by (2.1))}. \end{aligned}$$

Thus, letting $l \to \infty$, and then $k \to \infty$, the Monotone Convergence Theorem yields (a).

(b) If $B \subset X$ is open, its characteristic (or indicator) function is l.s.c. and of course nonnegative. Hence, (b) follows from (a). □

Proof of Theorem 2.1. For every measurable set $B \in \mathcal{B}$ and $\epsilon > 0$ there is an open set G_ϵ that contains B and $\mu(G_\epsilon - B) < \epsilon$ (see e.g. [2], p. 41). In particular, if B is a μ-null set, i.e., $\mu(B) = 0$ (and hence $\lambda_n(B) = 0\ \forall n$), we have $\mu(G_\epsilon) = \mu(G_\epsilon - B) < \epsilon$. Moreover, by part (b) in Lemma 3.1

$$\liminf_n \lambda_n(G_\epsilon) \geq \lambda(G_\epsilon) \geq \lambda(B).$$

Therefore, letting $\epsilon \downarrow 0$ we obtain $\lambda(B) = 0$. □

Proof of Theorem 2.2. Consider first the case $p = 1$. From $\liminf_n ||u_n||_1 \leq M$ and Lemma 3.1(b) (with $B := X$), it immediately follows that $\lambda(X) < \infty$ so that $u \in L_1$.

Consider now the case $p > 1$ and let q be the exponent conjugate to p. Then the condition (2.2) holds if uv is in L_1 for every $v \in L_q$ (see e.g. [5], p. 133). In turn, for this to be true it suffices to show that (as $u \geq 0$) there is a constant M such that

$$\int uv d\mu \leq M ||v||_q\ \forall v \in L_q^+. \tag{3.1}$$

Moreover, since (for $1 \leq q < \infty$) the class of continuous funtions in L_q is dense in L_q (see e.g. [2], p. 92) it suffices to prove (3.1) for all <u>continuous</u> functions v in L_q^+. (Indeed, suppose the latter holds, and let $v_m \in L_q^+$ be a sequence of continuous functions converging in the L_q-norm to $v \in L_q^+$. Let v_{m_i} be a subsequence converging to v μ-a.e. Then

$$\begin{aligned} M||v||_q = M \lim_i ||v_{m_i}||_q &\geq \lim_i \int u v_{m_i} d\mu \\ &\geq \int uv d\mu \quad \text{(by Fatou's Lemma)}; \end{aligned}$$

i.e., $v \in L_q^+$ satisfies (3.1).) Now, let $v \in L_q^+$ be a continuous function. Hence, (2.1) and Lemma 3.1(a) yield:

$$\begin{aligned} \int uv d\mu = \int v d\lambda &\leq \liminf_n \int v d\lambda_n \\ &= \liminf_n \int v u_n d\mu \\ &\leq \liminf_n ||u_n||_p ||v||_q \qquad \text{(Hölder's inequality)} \\ &\leq M||v||_q, \end{aligned}$$

with M as in Theorem 2.2. As v was an arbitrary continuous function in L_q^+ we obtain (3.1), hence (2.2). □

References

1. P. Billingsley, *Convergence of Probability Measures*, Wiley, New York, 1968. MR **38:**1718
2. J.L. Doob, *Measure Theory*, Springer-Verlag, New York, 1994. MR **95c:**28001
3. O. Hernández-Lerma and J.B. Lasserre, *Existence of solutions to the Poisson equation in L^p spaces*, LAAS Technical report 95045, Toulouse, 1995.
4. H.L. Royden, *Real Analysis*, 3rd Edition, Macmillan, New York, 1988. MR **90g:**00004
5. W. Rudin, *Real and Complex Analysis*, 3rd edition, McGraw-Hill, New York, 1986. MR **87f:**00009

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