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COVERING BY COMPLEMENTS OF SUBSPACES, II

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ABSTRACT. Let V be an n-dimensional vector space over an algebraically closed field K. Define $\gamma(k,n,K)$ to be the least positive integer t for which there exists a family $E_1, E_2, \dots, E_t$ of k-dimensional subspaces of V such that every $(n-k)$-dimensional subspace F of V has at least one complement among the E_i's. Using algebraic geometry we prove that $\gamma(k,n,K) = k(n-k)+1$.

1. Introduction

Take $V = V(n,K)$ to be an n-dimensional vector space over the algebraically closed field K. As usual a subspace F of V is a *complement* of the subspace E of V if $V = E \oplus F$, i.e., if $E + F = V$ and $E \cap F = \{0\}$. We let $c(E)$ denote the set of all complements of E in V and we write $G(k,n)$ for the set of all k-subspaces (= k-dimensional subspaces) of V. If $E \in G(k,n)$ then $c(E) \subseteq G(n-k,n)$. Define $\gamma(k,n,F)$ to be the least positive integer t such that there exist k-subspaces $E_1, E_2, \dots, E_t$ of V satisfying

$$c(E_1) \cup c(E_2) \cup \cdots \cup c(E_t) = G(n-k,n). \tag{1}$$

If (1) holds we say that all $(n-k)$-subspaces of V are *covered* by the E_i's.

In [1] we studied this problem for an arbitrary field K. Among other things we showed that in general $\gamma(k,n,K)$ depends on the field K. In particular, we showed that $\gamma(2,4,K)$ is 5 if K is quadratically closed and is 4 otherwise. We conjectured that $\gamma(k,n,K) = k(n-k)+1$ if K is algebraically closed. Here we prove this conjecture using results from algebraic geometry.

2. The lower bound $k(n-k)+1 \le \gamma(k,n,K)$

Let $\Lambda^k(V)$ denote the k-vectors in the exterior algebra $\Lambda(V)$ of V. We let $D(k,n)$ denote the set of all non-zero decomposable k-vectors $\alpha = v_1 \wedge v_2 \wedge \cdots \wedge v_k$ where $v_1, v_2, \dots, v_k$ are linearly independent vectors in V. Let $\langle\alpha\rangle$ denote the 1-dimensional subspace of $\Lambda^k(V)$ generated by α and write

$$\overline{D(k,n)} = \{\langle\alpha\rangle | \alpha \in D(k,n)\}. \tag{2}$$

If $v_1, v_2, \dots, v_k$ is a basis for $E \in G(k,n)$, then the mapping $E \mapsto \langle v_1 \wedge \cdots \wedge v_k\rangle$ is a bijection from $G(k,n)$ to $\overline{D(k,n)}$. It is well-known that this gives $G(k,n)$ the

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structure of an irreducible projective variety (the *Grassmannian*) of dimension $k(n-k)$ in $\mathbb{P}^N = \mathbb{P}(\Lambda^k(V))$ where $N = \binom{n}{k} - 1$. We identify $G(k,n)$ with $\overline{D(k,n)}$.

Now for any positive integer t let $G(k,n)^t$ be the product variety of $G(k,n)$ with itself t times. Let $E = (E_1, \ldots, E_t) \in G(k,n)^t$. For each i let $E_i = \langle \epsilon_i \rangle$ for some decomposable $\epsilon_i \in \Lambda^k(V)$. Define the mappings:

$$\varphi_i : \Lambda^{n-k}(V) \to \Lambda^n(V) \qquad \text{by} \qquad \varphi_i(\xi) = \epsilon_i \wedge \xi$$

for $i = 1, \ldots, t$ and let

$$\mathcal{K}(E) = ker(\varphi_1) \cap ker(\varphi_2) \cap \cdots \cap ker(\varphi_t).$$

Note that $\mathcal{K}(E)$ is a subspace of $\Lambda^{n-k}(V)$.

Lemma 1. *For $E \in G(k,n)^t$ the following two conditions are equivalent:*
(a) $c(E_1) \cup c(E_2) \cup \cdots \cup c(E_t) = G(n-k,n)$,
(b) $D(n-k,n) \cap \mathcal{K}(E) = \emptyset$.

Proof. This is an immediate consequence of the fact that if $F = \langle \alpha \rangle \in G(n-k,n)$ for some $\alpha \in D(n-k,n)$, then $E_i \cap F = \{0\}$ if and only if $\epsilon_i \wedge \alpha \neq 0$. $\square$

Lemma 2. *If $\gamma(k,n,K) = t$ and $E = (E_1, \ldots, E_t) \in G(k,n)^t$ satisfies*

$$c(E_1) \cup c(E_2) \cup \cdots \cup c(E_t) = G(n-k,n),$$

then

$$dim(\mathcal{K}(E)) = \binom{n}{k} - t.$$

Proof. Since φ_i is a linear mapping from the $\binom{n}{k}$-dimensional vector space $\Lambda^{n-k}(V)$ to the 1-dimensional vector space $\Lambda^n(V)$, it suffices to show that the mappings

$$\varphi_i \in hom(\Lambda^{n-k}(V), \Lambda^n(V)), \qquad i \in \{1, \ldots, t\},$$

are linearly independent. To see this we first note that the elements ϵ_i are linearly independent in $\Lambda^{n-k}(V)$. Suppose not; then we can assume that $\epsilon_t = \sum_{i=1}^{t-1} a_i \epsilon_i$. It follows that $\bigcap_{i=1}^{t} ker(\varphi_i) = \bigcap_{i=1}^{t-1} ker(\varphi_i)$. This implies by Lemma 1 that

$$c(E_1) \cup c(E_2) \cup \cdots \cup c(E_{t-1}) = G(n-k,n)$$

and hence $\gamma(k,n,K) \leq t-1$, a contradiction. Now assume that the mappings $\varphi_1, \ldots, \varphi_t$ are linearly dependent. Say, $\sum_{i=1}^{t} a_i \varphi_i = 0$. This means that for all $\xi \in \Lambda^{n-k}(V)$ we have $0 = \sum_{i=1}^{t} a_i(\epsilon_i \wedge \xi) = (\sum_{i=1}^{t} a_i \epsilon_i) \wedge \xi$. So it suffices to observe that if $\delta \in \Lambda^k(V)$ and if $\delta \wedge \xi = 0$ for all $\xi \in \Lambda^{n-k}(V)$ then $\delta = 0$. $\square$

Lemma 3. *If K is any algebraically closed field, then*

$$k(n-k) + 1 \leq \gamma(k,n,K).$$

Proof. Suppose $\gamma(k,n,K) = t \leq k(n-k)$. Then there exists $E = (E_1, \ldots, E_t) \in G(k,n)^t$ such that $c(E_1) \cup \cdots \cup c(E_t) = G(n-k,n)$. So by Lemmas 1 and 2 there is a linear subspace $\mathcal{K}(E)$ of $\Lambda^{n-k}(V)$ such that $D(k,n) \cap \mathcal{K}(E) = \emptyset$ and $\mathcal{K}(E)$ has affine dimension $\binom{n}{k} - t$ which is at least $\binom{n}{k} - k(n-k)$. Let $\mathcal{K}'$ denote the corresponding projective subspace of $\mathbb{P}(\Lambda^{n-k}(V))$. Then $\mathcal{K}' \cap G(n-k,n) = \emptyset$. But using projective dimensions we have [3, Proposition 11.4]

$$\begin{aligned} dim(\mathcal{K}') + dim(G(n-k,n)) &\geq \binom{n}{k} - k(n-k) - 1 + k(n-k) \\ &\geq \binom{n}{k} - 1 = dim(\mathbb{P}(\Lambda^{n-k}(V))) \end{aligned}$$

and it follows that $\mathcal{K}' \cap G(n-k,n) \neq \emptyset$ which is a contradiction. □

3. The upper bound $\gamma(k,n,K) \leq k(n-k)+1$

Lemma 4. *If K is algebraically closed, then*

$$\gamma(k,n,K) \leq k(n-k)+1.$$

Proof. Let $\nu = k(n-k)$ denote the dimension of $G(k,n)$ (and $G(n-k,n)$) as a projective variety. Let

$$A = G(k,n)^{\nu+1}.$$

Then A is a projective variety of dimension $\nu(\nu+1)$. For every $F \in G(n-k,n)$ define

$$B(F) = \{E \in G(k,n) | E \cap F \neq 0\}.$$

Now B(F) is an irreducible projective variety with

$$dim(B(F)) = \nu - 1.$$

For $F \in G(n-k,n)$ define

$$C(F) = B(F)^{\nu+1}.$$

Then

$$dim(C(F)) = (\nu+1)(\nu-1) = \nu^2 - 1.$$

Now set

$$C = \bigcup_{F \in G(n-k,n)} C(F).$$

Note that if C is properly contained in A, then there exists $E = (E_1, \ldots, E_{\nu+1}) \in A - C$. Then for all $F \in G(n-k,n)$ we have $E \notin C(F)$ so there must exist an index $i \in \{1, \ldots, \nu+1\}$ such that $E_i \cap F = 0$. Hence $c(E_1) \cup \cdots \cup c(E_{\nu+1}) = G(n-k,n)$ and so $\gamma(k,n,K) \leq \nu+1$, as desired. So it remains only to show that C is properly contained in A. In fact we claim that C is a variety of dimension at most $dim(A) - 1 = \nu^2 + \nu - 1$.

To complete the proof we fix $F_0 \in G(n-k,n)$ and consider the projective variety

$$D := C(F_0) \times PGL_n(K).$$

We note that

$$dim(D) = dim(C(F_0)) + dim(PGL(n,K)) = \nu^2 - 1 + n^2 - 1.$$

An element M of $PGL(n,K)$ induces a linear automorphism of $\mathbb{P}(\Lambda^k(V))$ which induces in turn an automorphism of $G(k,n)$. Abusing notation we write $U \mapsto MU$

to indicate the latter automorphism. Now we define $\varphi : D \to C$ as follows: For $(E, M) \in D$ set

$$\varphi(E, M) = (ME_1, ME_2, \ldots, ME_{\nu+1}).$$

Clearly φ is a regular surjection. Hence by [3, Theorem 11.12]

$$dim(D) = dim(C) + \mu$$

where

$$\mu = min\{dim(\varphi^{-1}(E'))\}, \quad E' \in C.$$

This shows that

$$dim(C) = \nu^2 - 1 + n^2 - 1 - \mu.$$

So to prove that $dim(C) \leq \nu^2 + \nu - 1$ it suffices to prove that $n^2 - \nu - 1 \leq \mu$. To see this consider the subset $G(F)$ of $PGL_n(K)$ whose elements map the fixed $(n-k)$-subspace F_0 to the $(n-k)$-space F. It is easy to see that $dim(G(F)) = n^2 - \nu - 1$. Now if $E' = (E'_1, \ldots, E'_{\nu+1}) \in C(F) \subseteq C$ then for each $M \in G(F)$ we have

$$(M^{-1}E', M) = (M^{-1}E'_1, \ldots, M^{-1}E'_{\nu+1}, M) \in \varphi^{-1}(E').$$

The mapping $M \mapsto (M^{-1}E', M)$ is a regular injection from $G(F)$ into the fiber $\varphi^{-1}(E')$. It follows that each fiber has dimension at least that of $G(F)$ and this completes the proof. □

Remarks. 1. The above proof shows that almost all $(E_1, \ldots, E_{\nu+1}) \in G(k, n)^{\nu+1}$ satisfy

$$c(E_1) \cup c(E_2) \cup \cdots \cup c(E_{\nu+1}) = G(n-k, n)$$

since the complement C of the set of such $(\nu+1)$-tuples forms a variety of dimension smaller than $dim(G(k, n)^{\nu+1})$

2. As shown in [1] $\gamma(2, 4, K) = 4$ when K is not quadratically closed. So the lower bound $\gamma(k, n, K) \geq k(n-k) + 1$ proved here for algebraically closed fields will not hold in general. On the other hand, we suspect that the upper bound $\gamma(k, n, K) \leq k(n-k) + 1$ does hold for arbitrary fields. In fact we have verified this for finite fields of sufficiently large order using counting arguments [2]. However, as the referee pointed out it is slightly worrying that the conjecture fails in the "thin" case, that is, if we replace n-space by n-set, k-subspace by k-subset and vector space complement by set complement. However, the upper bound of $\binom{n}{k}$ given in [1] holds in both cases.

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