

SOME RELATIONS BETWEEN NÖRLUND AND ABEL SUMMABILITY

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ABSTRACT. The definition of generalised Abel summability is extended to positive orders and a definition of strong generalised Abel summability is introduced. A result of Jurkat and Peyerimhoff concerning the implication between Nörlund and generalised Abel summabilities is extended to positive integral order. Analogous results for the corresponding strong methods are also given.

1. INTRODUCTION

Let (N, p) denote the Nörlund method of summability of a sequence (s_n) of real or complex numbers, and (A_0^*) the generalised Abel summability method. Concerning the implication between these methods of summability, Jurkat and Peyerimhoff proved the following result [5, Lemma 1]:

Theorem A. *If (N, p) is real and regular, and $s_n \rightarrow s(N, p)$, then $s_n \rightarrow s(A_0^*)$.*

In the case when $p_0 > 0$, $p_n \geq 0$, (N, p) is regular, Zygmund and, independently, Silverman and Tamarkin, proved that $s_n \rightarrow s(N, p)$ implies $s_n \rightarrow s(A_0^*)$, where (A_0^*) is a slightly more restrictive method than that considered in Theorem A (see Hardy's book [4, Theorem 18] for details).

In a recent note [8], Sarigöl considered the same question. His result is incorrect however, because he assumed that the quotient of two power series each with unit radius of convergence is a power series also having unit radius of convergence. This error has appeared before in related circumstances (see for example [7], where results for absolute summability, analogous to those considered here, are obtained).

The object of this note is to extend Theorem A so as to obtain conditions for which (N, p) summability implies (A_μ^*) summability, where μ is a positive integer (see §2 below, for relevant definitions). The results enable deduction of implications between the corresponding strong methods of summability.

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2. NOTATION AND DEFINITIONS

Denote by (p_n) a sequence of real numbers, and write $P_n = \sum_{r=0}^n p_r$, $P_n^{(1)} = \sum_{r=0}^n P_r$, etc. If

$$\frac{\sum_{v=0}^n p_{n-v} s_v}{P_n} \rightarrow s \quad \text{as } n \rightarrow \infty,$$

then the sequence (s_n) is said to be Nörlund summable to s , and we write $s_n \rightarrow s(N, p)$. In order for the (N, p) method to be regular, that is, for $s_n \rightarrow s$ to imply $s_n \rightarrow s(N, p)$, it is necessary and sufficient that $p_n = o(P_n)$ and $\sum_{v=0}^n |p_v| = O(|P_n|)$. If

$$\sum_{n=0}^N |p_n| \left| \frac{\sum_{v=0}^n p_{n-v} a_v}{P_n} - s \right|^\lambda = o(P_N),$$

then (s_n) is said to be strongly Nörlund summable to s with index $\lambda > 0$, and we write $s_n \rightarrow s[N, p]_\lambda$ (see [2]). In what follows, $\sum s_n x^n$ denotes $\sum_{n=0}^\infty s_n x^n$, etc.

A sequence (s_n) is said to be generalised Abel summable of order μ to s , or briefly (A_μ^*) summable to s , and we write $s_n \rightarrow s(A_\mu^*)$, $\mu > -1$, if $(1-x)^{\mu+1} \sum \binom{n+\mu}{n} s_n x^n$ has a positive radius of convergence and defines an analytic function $\sigma_\mu(x)$, regular in the unit disc except for poles, none of which lies on the interval $1-\varepsilon < x < 1$ for an arbitrarily small fixed positive ε , and $\sigma_\mu(x) \rightarrow s$ when $x \rightarrow 1$ —through real values. This definition is based on the (A_μ) method of Borwein [1]. The proof of Theorem A in [5] involves a similar definition to (A_μ^*) when $\mu = 0$. There, poles are allowed anywhere in the unit disc but regularity is restricted to $1-\varepsilon < x < 1$. However, as the proof makes clear, poles cannot occur on $1-\varepsilon < x < 1$, so that the (A_0^*) method in [5] implies our (A_0^*) method, when considering implications between (N, p) and (A^*) . A slightly more restrictive definition than (A_μ^*) above, excludes poles from the interval $0 \leq x < 1$, but is otherwise identical. The case $\mu = 0$ is that alluded to after Theorem A in §1.

In order to define strong generalised Abel summability $[A_\mu^*]_\lambda$, $\lambda \geq 1$, it is convenient to make a change of variable by writing $x = \frac{y}{1+y}$. Now if $s_n \rightarrow s(A_\mu^*)$ and $\int_M^Y |y\sigma'(y)|^\lambda dy = o(Y)$ as $Y \rightarrow \infty$, then (s_n) is strongly generalised Abel summable of order μ with index λ to the sum s , or $s_n \rightarrow s[A_\mu^*]_\lambda$. Here, M is an arbitrarily large fixed positive number and $\sigma'(y)$ denotes the derivative of $\sigma(y)$ with respect to y , where $\sigma(y) = \sigma_\mu(\frac{y}{1+y})$. This definition is based on the $[A_{\mu-1}]_\lambda$ method of Mishra [6, Theorem 4]. The more restrictive definition of $[A_\mu^*]_\lambda$, where poles are excluded from $0 \leq x < 1$ in (A_μ^*) , is obtained by replacing the lower limit of integration M by zero, and using the appropriate definition of (A_μ^*) .

3. RESULTS AND PROOFS

Theorem 1. *Let (N, p) be real and regular, and $n|p_n| = O(|P_n|)$. Then $s_n \rightarrow s(N, p)$ implies $s_n \rightarrow s(A_1^*)$.*

For the proof, we require two lemmas.

Lemma 1. *Let $p_n \geq 0$ and $np_n = O(P_n)$. Then $nP_n = O(P_n^{(1)})$.*

Proof. Since $np_n \leq MP_n$, it follows that

$$MP_n^{(1)} \geq \sum_{r=0}^n rp_r = nP_n - \sum_{r=0}^{n-1} P_r \geq nP_n - P_n^{(1)},$$

from which the conclusion follows. (I am grateful to the referee for pointing out this proof, which is simpler than my original.)

Lemma 2. *Let (p_n) be a sequence of real numbers with (N, p) real and regular, and $n|p_n| = O(|P_n|)$. Then $n|P_n| = O(|P_n^{(1)}|)$.*

Proof. Since (N, p) is regular, $p_n = o(P_n)$ and hence $\frac{P_{n-1}}{P_n} \rightarrow 1$. Thus (P_n) is a sequence of ultimately one-signed terms, and, without loss of generality, we may take this sign as positive. Also, since $\sum_{r=0}^n |p_r| = O(|P_n|)$, then $P_n^{(1)} \rightarrow \infty$. Write $P_n^{(1)*} = \sum_{r=0}^n |P_r|$, $P_n^* = \sum_{r=0}^n |p_r|$. Then $n|p_n| = O(|P_n|)$ implies $n|p_n| = O(P_n^*)$ which implies $nP_n^* = O(P_n^{(1)*})$, by Lemma 1. Since $P_n^{(1)} \rightarrow \infty$ and only a finite number of the P_n are negative, $P_n^{(1)*} = O(P_n^{(1)})$ so that $n|P_n| \leq nP_n^* = O(P_n^{(1)})$.

Proof of Theorem 1. Suppose $s_n \rightarrow s(N, p)$. Then $\sum(n+1)s_nx^n$ has a positive radius of convergence and, inside the circle of convergence,

$$\begin{aligned} (1-x)^2 \sum (n+1)s_nx^n &= (1-x) \sum \{(n+1)s_n - ns_{n-1}\}x^n \\ &= (1-x) \sum s_nx^n + (1-x) \sum na_nx^n. \end{aligned}$$

By Theorem A, $s_n \rightarrow s(A_0^*)$, so that it suffices to prove $na_n \rightarrow 0(A_0^*)$. Writing $a(x) = \sum a_nx^n = \frac{T(x)}{P(x)}$, where $T(x) = \sum P_nt_nx^n$, and t_n is the (N, p) transform of (s_n) , we have, inside the circle of convergence,

$$(1-x) \sum na_nx^n = (1-x)xa'(x) = (1-x)x \left\{ \frac{T'(x)}{P(x)} - \frac{T(x)P'(x)}{[P(x)]^2} \right\};$$

the primes denote differentiation with respect to x . The last expression equals

$$\begin{aligned} (1-x) &\left\{ \frac{\sum nP_nt_nx^n}{P(x)} - \frac{\sum P_nt_nx^n \sum nP_nx^n}{[P(x)]^2} \right\} \\ &= \frac{1}{\sum P_n^{(1)}x^n} \left\{ \sum nP_nt_nx^n - \sum P_nt_nx^n \frac{\sum nP_nx^n}{P(x)} \right\} \\ &= \frac{1}{\sum P_n^{(1)}x^n} \sum \left\{ nP_nx^n - \frac{P_nx^n}{P(x)} \sum nP_nx^n \right\} t_n \equiv \sum c_n(x)t_n \end{aligned}$$

say, where

$$(3.1) \quad c_n(x) = \frac{nP_nx^n - P_nx^n \sum nP_nx^n / P(x)}{\sum P_n^{(1)}x^n}.$$

We wish to show that when $t_n \rightarrow s$, $\sum c_n(x)t_n \rightarrow 0$ as $x \rightarrow 1-$. This is a sequence to function transformation, and using a suitable modification of Theorem 5 in [4], the necessary and sufficient conditions for the desired property to hold are (i) $\sum |c_n(x)| < H$, where H is positive constant independent of x for $0 < x_0 \leq x < 1$, (ii) $\sum c_n(x) \rightarrow 0$ as $x \rightarrow 1-$, and (iii) $c_n(x) \rightarrow 0$ as $x \rightarrow 1-$, for each n .

For (i), we require

$$(3.2) \quad \sum |c_n(x)| \leq \frac{\sum n|P_n|x^n}{|\sum P_n^{(1)}x^n|} + \frac{\sum |P_n|x^n |\sum nP_nx^n|}{|\sum P_nx^n| |\sum P_n^{(1)}x^n|} \leq H$$

when $0 < x_0 \leq x < 1$. As in the proof of Lemma 2, (P_n) must be ultimately one-signed, and we can take this sign as positive. Thus $(P_n^{(1)})$ is also ultimately positive, and since $P_n^{(1)} \rightarrow \infty$, the contribution of the negative terms to the sums will not affect the boundedness or otherwise of (3.2) for values of x sufficiently near 1. Thus the modulus signs in (3.2) can be dropped, and (3.2) holds provided $nP_n = O(P_n^{(1)})$, which follows from Lemma 2. Plainly, (ii) holds, since $\sum c_n(x)$ vanishes identically in this case. For (iii), we have from (3.1) that, for each n , $\lim_{x \rightarrow 1^-} nP_nx^n = nP_n$, $\lim_{x \rightarrow 1^-} P_nx^n = P_n$ are both finite, but both $\sum P_n^{(1)}x^n$ and $P(x)$ tend to infinity as $x \rightarrow 1^-$. Finally,

$$\sup_{0 \leq x < 1} \frac{|\sum nP_nx^n|}{\sum P_n^{(1)}x^n} < K,$$

a positive constant, provided $nP_n = O(P_n^{(1)})$, by the same argument as in (i). This completes the proof.

Remarks. 1. The reader should have no difficulty in formulating and proving the analogous result when (p_n) is positive and (A_1^*) refers to the method in which poles are excluded from $0 \leq x < 1$. We shall take such modifications for granted from now on.

2. With the same hypotheses, Theorem 1 can be extended to the case when (N, p) implies (A_μ^*) where μ is a positive integer. The proof involving induction on μ follows the same lines as that of Theorem 1, but requires μ differentiations of the quotient $\frac{T(x)}{P(x)}$. The details, while straightforward in principle, are somewhat involved in practice, and are consequently omitted.

3. The general case, when μ is a positive (non-integral) number, involves the difficulty that while $(A_\lambda) \Rightarrow (A_\mu)$ for $\lambda > \mu > -1$ [1, Theorem 2], it is not necessarily true that $(A_\lambda^*) \Rightarrow (A_\mu^*)$. The problem is that while $s_n \rightarrow s(A_\lambda^*)$, the function $\sigma_\mu(x)$ can have branch points in the unit circle, so that the definition of (A_μ^*) no longer holds. I have not been able to solve this problem, and suggest that if the implication can be proved at all, it will require a different method of proof from the above.

4. To obtain relations between strong Nörlund and strong Abel summability, we apply the analogue of an argument given by Flett [3, Theorem 4(i)] and consider the sequence of implications: $[N, p]_\lambda \Rightarrow (N, p) \Rightarrow (A_\mu^*) \Rightarrow [A_\mu^*]_\lambda$, where the λ 's (≥ 1), need not be equal.

The first implication holds under conditions given in [2, Theorems 6 and 7], the second is considered above, and the third follows from the relation

$$y\sigma'_\mu(y) = (\mu + 1)[\sigma_{\mu+1}(y) - \sigma_\mu(y)]$$

and regularity of the $(C, 1)$ mean for integrals. The result obtained is summarised in.

Theorem 2. *Let (N, p) be real and regular with $n|p_n| = O(|P_n|)$. Then $s_n \rightarrow s[N, p]_\lambda$ implies $s_n \rightarrow s[A_\mu^*]_\lambda$, $\lambda \geq 1$, μ a non-negative integer.*

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